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Unitary representations of $CM(3)$

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Abstract. Irreducible unitary representations of the group $CM(3)$, the 'three-dimensional collective motion group', which is the semidirect product of a six-dimensional Abelian group T_6 and $SL(3, \mathbb{R})$, are constructed. A countable basis is identified in the carrier space of each representation. On each $SL(3, \mathbb{R})$ orbit, elements of the Lie algebra $cm(3)$ are represented as differential operators. The relationship of the Bohr model and the $CM(3)$ model is discussed.

1. Introduction

The $CM(3)$ model [1–5] of nuclei is a microscopic formulation of Bohr's liquid drop model [6, 7] of nuclear collective quadrupole motion. Consider a nucleus consisting of A nucleons. Let x_{in} denote the ith component of the position of the nth nucleon in the Cartesian coordinate system. Let

$$Q_{ij} = \sum_{n=1}^{A} x_{in} x_{jn} \qquad i, j = 1, 2, 3. \tag{1.1}$$

These quadratic forms are decomposed into the monopole component

$$Q^0 = \tfrac{1}{3} \sum_{i=1}^{3} Q_{ii} \tag{1.2}$$

and the quadrupole components

$$\begin{aligned} Q^2_{\pm 2} &= \tfrac{1}{2}(Q_{11} - Q_{22}) \pm \mathrm{i} Q_{12} \qquad Q^2_{\pm 1} = \mp (Q_{13} \pm \mathrm{i} Q_{23}) \\ Q^2_0 &= \frac{1}{\sqrt{6}} (2Q_{33} - Q_{11} - Q_{22}). \end{aligned} \tag{1.3}$$

The Bohr model assumes that a nucleus is enclosed by a surface

$$R = R_0 \left(1 + \sum_{\mu=-2}^{2} \alpha_\mu \bar{Y}_{2\mu}(\theta, \phi) \right)$$

and the average $\langle \rho(n) \rangle$ of the density operator $\rho(n) = \Sigma_{n=1}^{A} \delta(n - n_n)$ is $3A/(4\pi R_0^3)$ if $|n| \leqslant R_0$ and 0 if $|n| > R_0$. If the parameters α_μ are assumed to be small, the averages of the quadrupole moments are

$$\langle Q^2_\mu\rangle = A\sqrt{\frac{3}{10\pi}}\,R_0^2\alpha_\mu \qquad \mu=0, \pm 1, \pm 2. \tag{1.4}$$

However, these expressions can be derived apart from the above assumptions. Let p_{in} be the momentum conjugate to x_{in} and

$$\mathrm{i}S_{jk} = \mathrm{i}\sum_{n=1}^{A}(x_{jn}p_{kn}+x_{kn}p_{jn}) \qquad j, k=1, 2, 3 \qquad j\neq k \tag{1.5a}$$

$$\mathrm{i}S_j = \mathrm{i}\sum_{n=1}^{A} x_{jn}p_{jn} \qquad j=1, 2, 3 \tag{1.5b}$$

$$\mathrm{i}L_k = \mathrm{i}\sum_{n=1}^{A}(x_{ln}p_{mn}-x_{mn}p_{ln}) \qquad (k, l, m \text{ is a cyclic permutation of } 1, 2, 3). \tag{1.5c}$$

Suppose that $|0\rangle$ denotes the state of a nucleus of angular momentum 0. Although $\langle 0|Q^2_\mu|0\rangle = 0$, the averages of quadrupole moments in the state

$$|\Phi\rangle = \exp\left(-\mathrm{i}\sum_{k=1}^{3}\xi_k S_k\right)|0\rangle$$

$\xi_k\in\mathbb{R}$, are not 0. That is,

$$\langle\Phi|Q^2_0|\Phi\rangle = q_0(\xi_i)\langle 0|Q^0|0\rangle \qquad \langle\Phi|Q^2_{\pm 2}|\Phi\rangle = \frac{q_2(\xi_i)}{\sqrt{2}}\langle 0|Q^0|0\rangle$$
$$\langle\Phi|Q^2_{\pm 1}|\Phi\rangle = 0 \tag{1.6}$$

where

$$q_0(\xi_i) = \frac{(2\mathrm{e}^{2\xi_3}-\mathrm{e}^{2\xi_1}-\mathrm{e}^{2\xi_2})}{\sqrt{6}} \qquad q_2(\xi_i) = \frac{(\mathrm{e}^{2\xi_1}-\mathrm{e}^{2\xi_2})}{\sqrt{2}}. \tag{1.7}$$

These expressions are derived from the commutation relations

$$\left[\mathrm{i}\sum_{k=1}^{3}\xi_k S_k,\, Q_{jl}\right] = (\xi_j+\xi_l)Q_{jl}$$

and the formula

$$\mathrm{e}^{Y}X\,\mathrm{e}^{-Y} = X+[Y,X]+\frac{1}{2!}[Y,[Y,X]]+\dots. \tag{1.8}$$

If we evaluate the mean values of the quadrupole moments in the rotated state $|\Psi\rangle = R(\phi, \theta, \psi)|\Phi\rangle$, where $R(\phi, \theta, \psi) = \exp(-\mathrm{i}\phi L_3)\exp(-\mathrm{i}\theta L_2)\exp(-\mathrm{i}\psi L_3)$, from equation (1.6) and $R^{-1}Q^2_\mu R = \Sigma^2_{\nu=-2} D^2_{\mu\nu}(\phi, \theta, \psi)Q^2_\nu$, we have

$$\langle\Psi|Q^2_\mu|\Psi\rangle = \langle 0|Q^0|0\rangle\bigg[D^2_{\mu,0}(\phi, \theta, \psi)q_0(\xi_i)$$
$$+\frac{1}{\sqrt{2}}(D^2_{\mu 2}(\phi, \theta, \psi)+D^2_{\mu,-2}(\phi, \theta, \psi))q_2(\xi_i)\bigg]. \tag{1.9}$$

(We adopt Bohr's D-function $D^J_{\mu\nu}(\phi, \theta, \psi) = \mathrm{e}^{\mathrm{i}\mu\phi}\,\mathrm{d}^J_{\mu\nu}(\theta)\,\mathrm{e}^{\mathrm{i}\nu\psi}$ throughout this paper.)

Expression (1.9) accords with expression (1.4) in the limit of small deformation. We may define the radius R_0 of the nuclei by $\langle 0|Q^0|0\rangle = R_0^2 A/5$. This is because, if the density of the nuclei is constant, the formula holds. Let $\beta_B = \sqrt{\Sigma_{\mu=-2}^{2} \alpha_\mu \bar{\alpha}_\mu}$ and $\beta \in \mathbb{R}$ be a parameter such that $\sqrt{2/3}\beta = \sqrt{5/4\pi}\beta_B$. If we set

$$\xi = \sum_{k=1}^{3} \xi_k \qquad \xi'_k = \left(\xi_k - \frac{\xi}{3}\right) = \sqrt{\tfrac{2}{3}}\, \beta \cos\left(\gamma - \frac{2\pi}{3} k\right) \qquad k = 1, 2, 3$$

and assume that β is small, we have

$$q_0(\xi_i) \approx 2\, e^{2\xi/3} \beta \cos \gamma \qquad q_2(\xi_i) \approx 2\, e^{2\xi/3} \beta \sin \gamma. \tag{1.10}$$

If we furthermore impose the condition of volume conservation $\xi = 0$, the right side of equation (1.9) becomes the product of $AR_0^2\sqrt{3/10\pi}$ and the well known expressions [6]

$$\alpha_\mu = \beta_B \left[D^2_{\mu,0}(\phi, \theta, \psi) \cos \gamma + \frac{1}{\sqrt{2}} (D^2_{\mu,2}(\phi, \theta, \psi) + D^2_{\mu,-2}(\phi, \theta, \psi)) \sin \gamma \right] \tag{1.11}$$

$$\mu = 0, \pm 1, \pm 2$$

of the deformation parameters α_μ. Thus we may regard $|\Psi\rangle$ as a deformed state in the sense of quantum mechanics.

However, even if the deformed states are identified, we do not know how the states vary as time passes. In the Bohr model, equations of motions for the parameters α_μ are established by assuming that a nuclei is a liquid drop whose motion is governed by classical fluid dynamics. In the $CM(3)$ model, we must look for a Hamiltonian which is an element of the universal enveloping algebra [8] generated by the operators Q_{ij}, S_{ij}, S_k and L_k. If we consider the correspondence of the parameters α_μ and the expectation values $\langle \Psi | Q^2_\mu | \Psi \rangle$, it is plausible to choose a Hamiltonian which accords with the Hamiltonian of the Bohr model in the limit of small deformation. Once a representation and Hamiltonian are chosen, the remaining problem is to identify a basis in the carrier space of the representation so that it becomes possible to diagonalize the Hamiltonian. It is the purpose of this article to construct irreducible representations of $CM(3)$ and identify bases for the carrier spaces.

The operators Q_{ij}, iS_{jk}, iS_k and iL_k form a real Lie algebra with respect to the commutation relation induced from the canonical commutation relations $[x_{jm}, p_{kn}] = i\delta_{jk}\delta_{mn}$. Let span$\{X_i\}$ denote the vector space generated by some vectors X_i. The Lie algebra $c = \mathrm{span}\{Q_{ij}, iS_{jk}, iS_k, iL_k\}$ is the semidirect sum [9] of the commutative ideal $t_6 = \mathrm{span}\{Q_{ij}\}$ and the subalgebra $g = \mathrm{span}\{iS_{jk}, iS_k, iL_k\}$. Denote $\frac{1}{3}\Sigma_{k=1}^{3} S_k$ by S^0 and $S_k - S^0$ by S_k^2, respectively. Let $sg = \mathrm{span}\{iS_{jk}, iS_k^2, iL_k\}$. Then sg is a subalgebra of g and $g = sg \oplus \mathrm{span}\{iS^0\}$, where iS^0 commutes with any element of sg. Since $\Sigma_{k=1}^{3} \xi_k S_k = (\Sigma_{k=1}^{3} \xi_k) S^0 + \Sigma_{k=1}^{3} \xi_k S_k^2$, removing S^0 from g and imposing the condition of volume conservation are equivalent. Here, we shall adopt the volume conservation condition and consider representations of the Lie algebra $cm(3) = t_6 \oplus sg$. By doing so, we can suppress scaling factors which make formulae onerous, do not lose any essential content of the theory and, furthermore, incorporating later volume variation by S^0 does not cause any difficulty.

2. Group $CM(3)$

In this section, we shall construct a topological group $CM(3)$ of the algebra $cm(3)$ from the following: (i) as is seen from the form of the wavefunction $|\Psi\rangle$, the deformation is generated by exponential functions of the elements of g; (ii) the representations of the Lie algebra are obtained from those of the group by differentiation. In general, a group generated by an algebra is more connotative. We need some additional assumptions to make a group of an algebra.

Instead of defining the domain and the range of the elements of $cm(3)$ in a Hilbert space, we shall proceed formally. For $\kappa=(\kappa_{11},\kappa_{12},\kappa_{13},\kappa_{22},\kappa_{23},\kappa_{33})\in\mathbb{R}^6$, let $U_k=\exp(\mathrm{i}\,\Sigma_{i\leqslant j}\,\kappa_{ij}Q_{ij})$, where the exponential is defined by the formal power series whose first term is the 'identity' I. Since Q_{ij}s commute mutually, $U_\kappa U_{\kappa'}=U_{\kappa+\kappa'}$ for $\kappa,\kappa'\in\mathbb{R}^6$. Let $\mathbf{0}=(0,0,0,0,0,0)$. As $U_0U_\kappa=U_\kappa U_0=U_\kappa$, $U_0=I$. From $U_\kappa U_{-\kappa}=I$, we have $U_\kappa^{-1}=U_{-\kappa}$. If we define an ε-neighbourhood of U_κ by $B(U_\kappa,\varepsilon)=\{U_{\kappa'}|\sqrt{\Sigma_{i\leqslant j}\,(\kappa_{ij}-\kappa'_{ij})^2}<\varepsilon\}$, then the collection $T_6=\{U_\kappa\,|\,\kappa\in\mathbb{R}^6\}$ becomes a six-dimensional Abelian group which is homeomorphic to $\mathbb{R}^6$.

Next we shall make a group of the Lie algebra sg and make it homoeomorphic to $SL(3,\mathbb{R})$. Let $X_{jk}=\mathrm{i}\,\Sigma_{n=1}^{A}\,x_{jn}p_{kn}$, $i,k=1,2,3$. Then $g=\mathrm{span}\{X_{ij}|i,j=1,2,3\}$. Let E_{ij} denote the matrix unit of 3×3 matrices whose (i,j) component is 1 and the other components are 0. A linear map $h:g\to gl(3,\mathbb{R})$ defined by $h(X_{ij})=-E_{ji}$ is an isomorphism, and so is the restriction $h|sg:sg\to sl(3,\mathbb{R})$.

The Lie algebra $sl(3,\mathbb{R})$ generates the group $SL(3,\mathbb{R})=\{g\in gl(3,\mathbb{R})|\det g=1\}$. However, the isomorphism of sg and $sl(3,\mathbb{R})$ does not *a priori* imply the homeomorphism of $SL(3,\mathbb{R})$ and a group generated by sg. Let $\|\varepsilon\|$ denote the Euclidean norm $\sqrt{\mathrm{Tr}(\varepsilon\varepsilon^{\mathrm{T}})}$ of $\varepsilon\in gl(3,\mathbb{R})$. Usually $sl(3,\mathbb{R})$ and $SL(3,\mathbb{R})$ are given the subspace topology of $\mathbb{R}^9$ induced by the Euclidean metric $d(\alpha,\beta)=\|\alpha-\beta\|$, where $\alpha,\beta\in sl(3,\mathbb{R})$ or $\alpha,\beta\in SL(3,\mathbb{R})$. In that topology, $sl(3,\mathbb{R})\cong\mathbb{R}^8$, where '$\cong$' means 'be homeomorphic to'.

If $\varepsilon\in sl(3,\mathbb{R})$, the exponential series $\exp(\varepsilon)=\Sigma_{n=0}^{\infty}\,\varepsilon^n/n!$ belongs to $SL(3,\mathbb{R})$. Let $\mathbf{0}$ and I denote zero and the identity of $sl(3,\mathbb{R})$ and $SL(3,\mathbb{R})$, respectively. Let $B(\mathbf{0},\log 2)=\{\varepsilon\in sl(3,\mathbb{R})|\ \|\varepsilon\|<\log 2\}$ and $W=\{\exp\varepsilon\,|\,\varepsilon\in B(\mathbf{0},\log 2)\}$. Then W is a neighbourhood of $I\in SL(3,\mathbb{R})$. On W, the logarithmic series $\log g=\Sigma_{n=1}^{\infty}\,(I-g)^n/n$ absolutely converges, and it holds that [10] $\exp(\log g)=g$. Since $\exp B(\mathbf{0},\log 2)$ and its inverse $\log|\,W$ are continuous, $B(\mathbf{0},\log 2)$ is homeomorphic to W.

As $W^{-1}=\{w^{-1}\,|\,w\in W\}=W$, any $g\in SL(3,\mathbb{R})$ can be represented a product of finitely many elements [11] of W, that is, $g=g_1\ldots g_n$ for some non-negative integer n and some $g_1,\ldots,g_n\in W$. If $g_i\in W$, $\varepsilon_i=\log g_i$ belongs to $B(\mathbf{0},\log 2)$ and therefore

$$g=\exp(\varepsilon_1)\ldots\exp(\varepsilon_n) \tag{2.1}$$

for some $\varepsilon_1,\ldots,\varepsilon_n\in B(\mathbf{0},\log 2)$.

For $\varepsilon=(\varepsilon_{ij})\in sl(3,\mathbb{R})$ let

$$U_\varepsilon=\exp\left(-\sum_{i,j=1}^{3}\varepsilon_{ij}\mathrm{X}_{ji}\right)$$

be a formal power series. Then $U_0=I$ and $U_\varepsilon^{-1}=U_{-\varepsilon}$. Let

$$W_{sg}^n=\{U_{\varepsilon_1}\ldots U_{\varepsilon_n}|\,\varepsilon_1,\ldots,\varepsilon_n\in sl(3,\mathbb{R})\}.$$

Then $SG=\bigcup_{n=1}^{\infty}W_{sg}^n$ becomes a group.

From the commutation relation

$$\left[\sum_{i,j=1}^{3} \varepsilon_{ij} X_{ji}, x_{kn}\right] = \sum_{i=1}^{3} \varepsilon_{kj} x_{jn} \qquad \mathrm{i}=1, 2, 3 \qquad n=1, 2, \ldots, A$$

and formula (1.8), we have

$$U_\varepsilon x_{in} U_\varepsilon^{-1} = \sum_{j=1}^{3} [\exp(-\varepsilon^{\mathrm{T}})]_{ji} x_{jn} \qquad i=1, 2, 3 \qquad n=1, 2, \ldots, A. \tag{2.2}$$

Let ϕ denote this map $U_\varepsilon \to \exp(-\varepsilon^{\mathrm{T}})$. Then the map $\phi: SG \to SL(3, \mathbb{R})$ satisfies $\phi(U_\varepsilon U_{\varepsilon'}) = \phi(U_\varepsilon)\phi(U_{\varepsilon'})$. Also it is surjective. The reason for this is, given $g \in SL(3, \mathbb{R})$, $g = \exp(\varepsilon_1) \ldots \exp(\varepsilon_n)$ for some $\varepsilon_1, \ldots, \varepsilon_n \in B(0, \log 2)$ and $\phi(U_{-\varepsilon_1^{\mathrm{T}}} \ldots U_{-\varepsilon_n^{\mathrm{T}}}) = g$.

Let Ker ϕ denote the kernel of ϕ. The kernel is not a one-point set. For example, elements $\exp(2m\pi\, \mathrm{i} L_k)$, where $k=1, 2, 3$ and $m \in \mathbb{Z}$, belong to Ker ϕ. By the homomorphism theorem of group theory, $SG/\mathrm{Ker}\ \phi$ is isomorphic to $SL(3, \mathbb{R})$. Let ψ denote this isomorphism. If we define $\psi^{-1}(V)$ is open in $SG/\mathrm{Ker}\ \phi$ if and only if V is open in $SL(3, \mathbb{R})$, then $SG/\mathrm{Ker}\ \phi$ becomes homeomorphic to $SL(3, \mathbb{R})$. We hereafter denote $SG/\mathrm{Ker}\ \phi$ by $SL(3, \mathbb{R})$.

For $g = \exp(\varepsilon_1) \ldots \exp(\varepsilon_n)$, denote the residue class of $U_{-\varepsilon_1^{\mathrm{T}}} \ldots U_{-\varepsilon_n^{\mathrm{T}}} \in SG$ by U_g. Also denote $(g^{\mathrm{T}})^{-1} = (g^{-1})^{\mathrm{T}}$ and its (i, j) component by g^* and g^*_{ij}, respectively. Then by equation (2.2) and the definition (1.1) of Q_{ij}, for $a \in SL(3, \mathbb{R})$,

$$Q'_{ij} = U_a Q_{ij} U_a^{-1} = \sum_{k,l=1}^{3} a^*_{ki} a^*_{lj} Q_{kl} = \sum_{k \leqslant l} A_{kl,ij}(a) Q_{kl} \tag{2.3}$$

where

$$A_{kl,ij}(a) = a^*_{ki} a^*_{lj} + a^*_{li} a^*_{kj} \qquad \text{if } k<l$$

$$A_{kk,ij} = a^*_{ki} a^*_{kj}.$$

By straightforward calculation, the determinant of the 6×6 matrix $(A_{kl,ij}(a))$ is shown to be $(\det a)^{-4}$, which is 1 for $\det a = 1$. If we arrange Q_{ij}s in the form of a 3×3 symmetric matrix,

$$Q = \begin{pmatrix} Q_{11} & Q_{12} & Q_{13} \\ Q_{12} & Q_{22} & Q_{23} \\ Q_{13} & Q_{23} & Q_{33} \end{pmatrix} \tag{2.4}$$

and denote $a^{-1} Q (a^{-1})^{\mathrm{T}}$ by $a^{-1} \cdot Q$ then equation (2.3) can be written as

$$Q' = U_a Q U_a^{-1} = a^{-1} \cdot Q. \tag{2.5}$$

Now, the collection $CM(3) = \{U_\kappa U_a \mid U_\kappa \in T_6, U_a \in SL(3, \mathbb{R})\}$ is shown to be the semidirect product group $T_6 \times_{\mathrm{sd}} SL(3, \mathbb{R})$. In fact, $I = U_0 U_1$ is the identity. For $\kappa \in \mathbb{R}^6$ and $a \in SL(3, \mathbb{R})$, let $a \cdot \kappa$ be defined by $\Sigma_{k \leqslant l} A_{ij,kl}(a) \kappa_{kl}$. Then $U_a U_\kappa U_a^{-1} = U_{a \cdot \kappa} \in T_6$. Therefore $U_a^{-1} U_\kappa^{-1} = U_{-a^{-1} \cdot \kappa} U_a^{-1} \in CM(3)$ is the inverse of $U_\kappa U_a$ and $U_\kappa U_a U_{\kappa'} U_{a'} = U_\kappa (U_a U_{\kappa'} U_a^{-1}) U_a U_{a'} = U_{\kappa' + a \cdot \kappa} U_{aa'} \in CM(3)$.

Since $\det(A_{ij,kl}(a))$ is not 0, the linear map $\kappa_{ij} \to \Sigma_{k\leq l} A_{ij,kl}(a)\kappa_{kl}$ is an automorphism of $\mathbb{R}^6 = \{(\kappa_{11}, \ldots, \kappa_{33})\}$. Thus $CM(3)$, a subspace of $\mathbb{R}^6 \times \mathbb{R}^9$, is a semidirect product [12] of T_6 and $SL(3, \mathbb{R})$.

3. $SL(3, \mathbb{R})$ orbits in $\mathbb{R}^6$ and the isotropy subgroups

Since T_6 is Abelian, its irreducible representations are one dimensional. For $U_\kappa \in T_6$, the map $x_{Q'}: U_\kappa \to \exp(\mathrm{i}\, \Sigma_{i\leq j} \kappa_{ij} Q'_{ij})$ is the representation labelled by $Q' = (Q'_{11}, Q'_{12}, \ldots, Q'_{33}) \in \mathbb{R}^6$. As usual, we give $\mathbb{R}^6$ the standard topology. In the bra–ket formalism [13], if we denote by $|Q'\rangle = |Q'_{11}, Q'_{12}, \ldots, Q'_{33}\rangle$ the simultaneous eigenstate of Q_{ij}s, then

$$U_\kappa |Q'\rangle = x_{Q'}(U_\kappa)|Q'\rangle \tag{3.1}$$

and $\{|Q'\rangle\}$ is the one-dimensional carrier space. If we operate with equation (2.3) on $|Q'\rangle$, then

$$U_a Q_{ij} U_a^{-1}|Q'\rangle = Q''_{ij}|Q'\rangle \qquad Q''_{ij} = \sum_{k\leq l} A_{kl,ij}(a) Q'_{kl} = (a^{-1} \cdot Q')_{ij} \tag{3.2}$$

where Q' denotes the matrix of the form of equation (2.4) whose components are real numbers. Since Q'_{ij}s are c-numbers,

$$Q_{ij} U_a^{-1}|Q'\rangle = Q''_{ij} U_a^{-1}|Q'\rangle \tag{3.3}$$

which implies that $U_a^{-1}|Q'\rangle$ is the eigenstate of the quadrupole moments Q_{ij} belonging to the eigenvalues Q''_{ij}. That is,

$$U_a^{-1}|Q'\rangle = c|Q''\rangle \qquad Q'' = a^{-1} \cdot Q' \tag{3.4}$$

where c is a function of $a \in SL(3, \mathbb{R})$ and Q', which will be suitably chosen in section 6. Equation (3.4) shows that $U_a \in SL(3, \mathbb{R})$ connects carrier spaces of the representations of T_6. However, not arbitrary $|Q'\rangle$ and $|Q''\rangle$ are connected by some U_a.

If we define a relation by

$$Q'' \sim Q' \qquad \text{if } Q'' = aQ'a^{\mathrm{T}} = a \cdot Q \quad \text{for some } a \in SL(3, \mathbb{R})$$

this relation is an equivalence relation. On each equivalence class, called an orbit, $SL(3, \mathbb{R})$ acts transitively. We shall list below all of these orbits. We hereafter denote the eigenvalues of the quadrupole moments by Q_{ij}, so far as no confusion arises. If $\det Q \neq \det Q'$ then Q and Q' belong to different orbits. Let

$$\Lambda = \{\mathrm{diag}(\lambda_1, \lambda_2, \lambda_3) | \lambda_i > 0, \lambda_1\lambda_2\lambda_3 = 1\} \subset SL(3, R).$$

Since $Q \in \mathbb{R}^6$ is a real symmetric matrix, there exist $\lambda \in \Lambda$ and $r \in SO(3)$ such that $(r\lambda)^{\mathrm{T}} Q r\lambda$ is equal to one of the following matrices:

$$\pm l\, \mathrm{diag}(1, 1, 1) \qquad \pm l\, \mathrm{diag}(-1, -1, 1) \tag{3.5}$$

where $l = |\det Q|^{1/3} \neq 0$,

$$\pm\mathrm{diag}(1, 1, 0) \qquad \mathrm{diag}(1, -1, 0) \qquad \pm\mathrm{diag}(0, 0, 1) \qquad \mathbf{0} = \mathrm{diag}(0, 0, 0). \tag{3.6}$$

That is, any Q is equivalent to one of the above matrices. Furthermore, by Sylvester's law of inertia [14], matrices given in equations (3.5) and (3.6) are non-equivalent, because they have different signatures. We shall call these matrices the origins of the orbits.

Thus, there are the following $SL(3, \mathbb{R})$ orbits:

$$S_1^{\pm}(l) = \{Q = \pm l r \lambda^2 r^{\mathrm{T}} | r \in SO(3), \lambda \in \Lambda\} \qquad l > 0$$

$$S_2^{\pm}(l) = \{Q = \pm l r \lambda^2 \operatorname{diag}(-1, -1, 1) r^{\mathrm{T}} | r \in SO(3), \lambda \in \Lambda\} \qquad l > 0$$

$$S_3^{\pm} = \{Q = \pm r \lambda^2 \operatorname{diag}(1, 1, 0) r^{\mathrm{T}} | r \in SO(3), \lambda \in \Lambda\}$$

$$S_4 = \{Q = \pm r \lambda^2 \operatorname{diag}(1, -1, 0) r^{\mathrm{T}} | r \in SO(3), \lambda \in \Lambda\}$$

$$S_5^{\pm} = \{Q = \pm r \lambda^2 \operatorname{diag}(0, 0, 1) r^{\mathrm{T}} | r \in SO(3), \lambda \in \Lambda\}$$

$$S_6 = \{\mathbf{0}\}.$$

It is natural to give all of these orbits the subspace topology of $\mathbb{R}^6$. Although $\mathbb{R}^6$ is the union of all of these orbits, since the inversion $Q \to -Q$ and the scale transformation $Q \to lQ (l > 0)$ are diffeomorphisms, there are only six orbits, $S_1^+ = S_1^+(1)$, $S_2^+ = S_2^+(1)$, S_3^+, S_4, S_5^+ and S_6, which may be mutually not homeomorphic. In fact, since the dimensions of S_6 and S_5^+ are 0 and 3, respectively, they are not mutually homeomorphic nor homeomorphic with the other orbits whose dimensions are 5.

Let O denote the origin of an $SL(3, \mathbb{R})$ orbit. The isotropy subgroup of $SL(3, \mathbb{R})$ with respect to the origin O is found by solving the equation $O = aOa^{\mathrm{T}}$, $a \in SL(3, \mathbb{R})$. We denote the isotropy subgroups of S_1^+, S_2^+, S_3^+, S_4, S_5^+ and S_6 by K_1, K_2, K_3, K_4, K_5 and K_6, respectively. Obviously $K_6 = SL(3, \mathbb{R})$ and $K_1 = SO(3)$. Let $C_x = \operatorname{diag}(1, -1, -1)$, which is the π-rotation about the x-axis. The remaining isotropy subgroups are listed below:

$$K_2 = SO(2, 1) = SO(2, 1)_0 \cup C_x SO(2, 1)_0$$

where $SO(2, 1)_0$ is the three-dimensional proper Lorentz group which is a normal subgroup of $SO(2, 1)$.

$$K_3 = M(2) \cup C_x M(2)$$

where

$$M(2) = \left\{ \left. \begin{pmatrix} \cos\theta & -\sin\theta & a \\ \sin\theta & \cos\theta & b \\ 0 & 0 & 1 \end{pmatrix} \right| a, b, \theta \in \mathbb{R} \right\}$$

is the two-dimensional Euclidean motion group [15], and is a normal subgroup of K_3.

$$K_4 = \left\{ \left. \begin{pmatrix} \cosh\tau & \sinh\tau & a \\ \sinh\tau & \cosh\tau & b \\ 0 & 0 & 1 \end{pmatrix} \right| a, b, \tau \in \mathbb{R} \right\}$$

which is denoted by $MH(2)$ in [15].

$$K_5 = SA(2) \cup C_x SA(2)$$

where

$$SA(2)=\left\{\begin{pmatrix} a & b & 0 \\ c & d & 0 \\ x & y & 1 \end{pmatrix}\middle| x, y, a, b, c, d\in\mathbb{R}, ad-bc=1\right\}$$

which is a normal subgroup of K_5. It will be appropriate to call $SA(2)$ the 'special two-dimensional affine motion group' because it is a semidirect product $T_2 \underset{\mathrm{sd}}{\times} SL(2, \mathbb{R})$ of

$$SL(2, \mathbb{R})=\left\{\begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{pmatrix}\middle| a, b, c, d\in\mathbb{R}, ad-bc=1\right\}$$

and an Abelian group

$$T_2=\left\{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ x & y & 1 \end{pmatrix}\middle| x, y\in\mathbb{R}\right\}.$$

Unitary representations of $SO(2.1)_0$, $M(2)$, $MH(2)$ are known [15], and those of $SA(2)$ can be obtained by the method of induced representation explained in section 6. Representations of K_2, K_3, K_4 and K_5 are obtained by the method of induced representation given in [16]. Representations of $K_6=SL(3, \mathbb{R})$ are given in [17].

4. Fundamental Groups of $SL(3, \mathbb{R})$ orbits

In order to discriminate the structures of the $SL(3, \mathbb{R})$ orbits, we consider their connectivity. It is known that S_1^+ is homeomorphic [18] to $\mathbb{R}^5=\{X\in gl(3, \mathbb{R})|X=X^{\mathrm{T}}, \operatorname{Tr} X=0\}$. That is, S_1^+ is simply connected. Let S and O be one of the $SL(3, \mathbb{R})$ orbits and its origin. If $Q\in S$, then $Q=r\lambda^2 O r^{\mathrm{T}}$, where $\lambda=(e^{\xi_1}, e^{\xi_2}, e^{\xi_3})$, $\xi_i\in\mathbb{R}$ and $\xi_1+\xi_2+\xi_3=0$. For $t\in[0, 1]$, let $\lambda(t)=(e^{t\xi_1}, e^{t\xi_2}, e^{t\xi_3})$ and $Q(t)=r\lambda(t)^2 O r^{\mathrm{T}}$. Then $Q(t)$, $t\in[0, 1]$ is a strong deformation retraction [19] of S and $M=\{Q=rOr^{\mathrm{T}}|r\in SO(3)\}$. The fundamental groups of S and M are isomorphic [19].

First we take up S_5^+. Let

$$h(\phi)=\begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad g(\theta)=\begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}.$$

Then any $r\in SO(3)$ is represented by the product $h(\phi)g(\theta)h(\psi)$, where ϕ, $\psi\in\mathbb{R}$ mod 2π and $\theta\in[0, \pi]$. The strong deformation retract of S_5^+ is $M_5=\{r\operatorname{diag}(0, 0, 1)r^{\mathrm{T}}|r\in SO(3)\}$. If $Q\in M_5$, then $Q=h(\phi)g(\theta)\operatorname{diag}(0, 0, 1)g^{\mathrm{T}}(\theta)h^{\mathrm{T}}(\phi)$. Let $x_1=\sin\theta\cos\phi$, $x_2=\sin\theta\sin\phi$, $x_3=\cos\theta$. Then the (i, j) component q_{ij} of Q is $x_i x_j$. Therefore, M_5 is the image set of the map $f: S^2\to\mathbb{R}^6$ defined by $f(x_1, x_2, x_3)_{ij}=x_i x_j$. According as $q_{11}\neq 0$, $q_{22}\neq 0$ and $q_{33}\neq 0$, we can take $(q_{12}/q_{11}, q_{13}/q_{11})$, $(q_{12}/q_{22}, q_{23}/q_{22})$ and $(q_{13}/q_{33}, q_{23}/q_{33})$ as local coordinates of M_5. Coordinate transformation functions are infinitely many times continuously differentiable.

By the map f, points (x_1, x_2, x_3) and $(-x_1, -x_2, -x_3)$ of S^2 are mapped to the same point of M_5. Conversely, $f^{-1}(Q)$ consist of two points. For example, suppose that

$q_{33} \neq 0$ and let $u_1 = q_{13}/q_{33}$ and $u_2 = q_{23}/q_{33}$. Then

$$f^{-1}(Q) = \left\{ \pm \left(\frac{u_1}{\sqrt{u_1^2+u_2^2+1}}, \frac{u_2}{\sqrt{u_1^2+u_2^2+1}}, \frac{1}{\sqrt{u_1^2+u_2^2+1}} \right) \right\}. \tag{4.1}$$

Thus, M_5 is a set consisting of the pairs of antipodal points of S^2. However, in order to infer that M_5 is the two-dimensional projective space P^2, we need to show that the quotient topology [20] of M_5 induced by f is the same as the given topology. It suffices to show that $f: S^2 \to \mathbb{R}^6$ is an open map. Let V be a subset of M_5 such that $U = f^{-1}(V)$ is open. Let $Q \in V$ and $x \in f^{-1}(Q)$. Since U is open, there exist a neighbourhood $E \subset U$ of x. As is seen from equation (4.1), if E is chosen sufficiently small then $x \in E$, $-x \notin E$, the restriction $f|E: E \to f(E)$ is bijective and $(f|E)^{-1}$ is continuous.

Therefore, $((f|E)^{-1})^{-1}(E) = f(E)$ is open. Then $f(E) \subset V$ is a neighbourhood of $Q \in V$. Thus V is open. The fundamental group of P^2 is the two-point group. Thus, M_5 and therefore S_5^+ are doubly connected.

Strong deformation retracts of S_2^+ and S_3^+ are $M_2 = \{r\,\mathrm{diag}(-1,-1,1)r^{\mathrm{T}} | r \in SO(3)\}$ and $M_3 = \{r\,\mathrm{diag}(1,1,0)r^{\mathrm{T}} | r \in SO(3)\}$, respectively. Because $\mathrm{diag}(-1-1,1) = -I + 2\,\mathrm{diag}(0,0,1)$ and $\mathrm{diag}(1,1,0) = I - \mathrm{diag}(0,0,1)$, M_2 and M_3 are homeomorphic to P^2. Thus, all of S_2^+, S_3^+ and S_5^+ are doubly connected.

The structure of S_4 is more complicated. Let $f: SO(3) \to \mathbb{R}^6$ be the map defined by $f(r) = r\,\mathrm{diag}(1,-1,0)r^{\mathrm{T}}$. Then $M_4 = f(SO(3))$. Let $D_2 = \{I, C_x, C_y, C_z\}$ be the dihedral group consisting of the identity and π-rotations about the x, y, z axes. If $C \in D_2$ then $f(rC) = f(r)$. Conversely, if $f(rC) = f(r)$ for $C \in SO(3)$ then $C \in D_2$. This is shown by solving the equation $C\,\mathrm{diag}(1,-1,0)C^{\mathrm{T}} = \mathrm{diag}(1,-1,0)$, $C \in SO(3)$. Thus, M_4 is the quotient set $SO(3)/D_2$. First we shall show that M_4 is a three-dimensional manifold embedded in $\mathbb{R}^6$. Let Δ_{ij} denote the (i,j) cofactor of $\det Q$. Since the eigenvalues of $Q \in M_4$ are $1, -1, 0$,

$$h_1(Q) = \mathrm{Tr}\, Q = 0 \qquad h_2(Q) = \sum_{i=1}^{3} \Delta_{ii} + 1 = 0 \qquad h_3(Q) = \det Q = 0.$$

If we define $h: \mathbb{R}^6 \to \mathbb{R}^3$ by $h(Q) = (h_1(Q), h_2(Q), h_3(Q))$, then $M_4 = h^{-1}(0)$. The derivative of h at Q is

$$\mathrm{d}h_Q = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ -q_{11} & -2q_{12} & -2q_{13} & -q_{22} & -2q_{23} & -q_{33} \\ \Delta_{11} & 2\Delta_{12} & 2\Delta_{13} & \Delta_{22} & 2\Delta_{23} & \Delta_{33} \end{pmatrix}$$

where h_1, h_2, h_3 and $q_{11}, q_{22}, \ldots, q_{33}$ are arranged vertically and horizontally, respectively. By elementary linear algebra, the rank of $\mathrm{d}h_Q$ is shown to be 3. Therefore, M_4 is a three-dimensional subspace [21] of $\mathbb{R}^6$, and three of the $q_{11}, q_{22}, \ldots, q_{33}$ serve as local coordinates.

Although we do not write down the lengthy algebraic expressions of $f^{-1}(Q)$, it is shown that if $r \in f^{-1}(Q)$, then $f^{-1}(Q) = \{r, rC_x, rC_y, rC_z\}$. In the same way as above, f can be shown to be an open map, and consequently $SO(3)/D_2$ in the quotient topology is homeomorphic to M_4.

Let $\mathrm{Ad}: SU(2) \to SO(3)$ be the adjoint representation [22] of $SU(2)$. Let $p = f \cdot \mathrm{Ad}$. Then $p: SU(2) \to M_4$ is shown to be an open and covering map [23]. Let $\tilde{D}_2 = \mathrm{Ad}^{-1}(D_2)$, then

$$\tilde{D}_2 = \left\{ \pm\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \pm\begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \pm\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \pm\begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix} \right\}.$$

Since $M_4 \cong SO(3)/D_2 \cong SU(2)/\tilde{D}_2$, and $SU(2)$ is simply connected, the fundamental group of M_4 is isomorphic [24]† to $\tilde{D}_2$. Thus M_4, and therefore S_4, is an eight-fold connected space. In conclusion, except for the possibility that S_2^+ and S_3^+ are homeomorphic, all orbits $S_1^+, S_2^+, S_3^+, S_4, S_5^+$ are not mutually homeomorphic.

5. Measures on $SL(3, \mathbb{R})$ orbits

Suppose that S is a k-dimensional surface in $\mathbb{R}^6$ and $(\alpha_1, \ldots, \alpha_k) \to Q \in S$ is its parametrization (differentiable surjection). Set $x_1 = Q_{11}$, $x_2 = \sqrt{2} Q_{12}$, $x_3 = \sqrt{2} Q_{13}$, $x_4 = Q_{22}$, $x_5 = \sqrt{2} Q_{23}$ and $x_6 = Q_{33}$. Then the measure on S induced by the Euclidean norm $\|Q\| = \sqrt{\operatorname{Tr} Q^2} = \sqrt{\Sigma_{i=1}^6 x_i^2}$, is

$$\mathrm{d}\mu(Q) = \sqrt{\sum_{i_1 < \ldots < i_k} \left| \frac{\partial(x_{i_1}, \ldots, x_{i_k})}{\partial(\alpha_1, \ldots, \alpha_k)} \right|^2} \, \mathrm{d}\alpha_1 \ldots \mathrm{d}\alpha_k. \tag{5.1}$$

With the aid of this formula, we can calculate the measures on any $SL(3, \mathbb{R})$ orbit. We shall suppress numeral factors of the measures which are absorbed in the normalization of wavefunctions.

Let $r(\phi, \theta, \psi) = h(\phi) g(\theta) h(\psi) \in SO(3)$. If $Q \in S_5^+$, then $Q(\phi, \theta, \nu) = r(\phi, \theta, \psi) \operatorname{diag}(0, 0, \nu) r^{\mathrm{T}}(\phi, \theta, \psi) = h(\phi) g(\theta) \operatorname{diag}(0, 0, \nu) g^{\mathrm{T}}(\theta) h^{\mathrm{T}}(\phi)$, where $\nu \in \mathbb{R}^+$. We can take ν, θ, ϕ as parameters of the orbit S_5^+. By equation (5.1),

$$\mathrm{d}\mu_5(Q) = \nu^2 \, \mathrm{d}\nu \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi. \tag{5.2}$$

If $Q \in S_3^+$ or S^4, then

$$Q(\phi, \theta, \psi, \lambda_1, \lambda_2) = r(\phi, \theta, \psi) \operatorname{diag}(\lambda_1, \pm\lambda_2, 0) r^{\mathrm{T}}(\phi, \theta, \psi) \qquad \lambda_1, \lambda_2 \in \mathbb{R}^+.$$

According as $Q \in S_3^+$ or $Q \in S_4$, equation (5.1) implies that

$$\mathrm{d}\mu_3(Q) = \lambda_1 \lambda_2 |\lambda_1 - \lambda_2| \, \mathrm{d}\lambda_1 \, \mathrm{d}\lambda_2 \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi \, \mathrm{d}\psi \tag{5.3}$$

$$\mathrm{d}\mu_4(Q) = \lambda_1 \lambda_2 (\lambda_1 + \lambda_2) \, \mathrm{d}\lambda_1 \, \mathrm{d}\lambda_2 \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi \, \mathrm{d}\psi. \tag{5.4}$$

The measures (5.2), (5.3) and (5.4) are quasi-invariant [25].

If $Q \in S_1^+$ or $Q \in S_2^+$, then

$$Q(\phi, \theta, \psi, \lambda_1, \lambda_2, \lambda_3) = r(\phi, \theta, \psi) \operatorname{diag}(\pm\lambda_1, \pm\lambda_2, \lambda_3) r^{\mathrm{T}}(\phi, \theta, \psi)$$

$$\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}^+ \qquad \lambda_1 \lambda_2 \lambda_3 = 1$$

and $\operatorname{Tr} Q^{-2} = \Sigma_{i=1}^3 1/\lambda_i^2$. If we introduce two independent variables $\varepsilon_0, \varepsilon_2 \in \mathbb{R}$ such that $\lambda_1 = \exp[2(\varepsilon_2/\sqrt{2} - \varepsilon_0/\sqrt{6})]$, $\lambda_2 = \exp[2(-\varepsilon_2/\sqrt{2} - \varepsilon_0/\sqrt{6})]$, $\lambda_3 = \exp(2\sqrt{2/3}\,\varepsilon_0)$, then, according as $Q \in S_1^+$ or $Q \in S_2^+$, quasi-invariant measures calculated from formula (5.1) are

$$\mathrm{d}\mu_1(Q) = \sqrt{\operatorname{Tr} Q^{-2}} \, |\lambda_1 - \lambda_2| \, |\lambda_2 - \lambda_3| \, |\lambda_3 - \lambda_1| \, \mathrm{d}\varepsilon_0 \, \mathrm{d}\varepsilon_2 \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi \, \mathrm{d}\psi$$

$$\mathrm{d}\mu_2(Q) = \sqrt{\operatorname{Tr} Q^{-2}} |\lambda_1 - \lambda_2| (\lambda_2 + \lambda_3)(\lambda_3 + \lambda_1) \, \mathrm{d}\varepsilon_0 \, \mathrm{d}\varepsilon_2 \sin\theta \, \mathrm{d}\theta \, \mathrm{d}\phi \, \mathrm{d}\psi.$$

However, on S_1^+ and S_2^+, there exist invariant measures induced from the measure $\mathrm{d}V = \mathrm{d}x_1 \, \mathrm{d}x_2 \ldots \mathrm{d}x_6$ of $\mathbb{R}^6$. The group $GL^+(3, \mathbb{R})$ is the direct product $\mathbb{R}^+ \times SL(3, \mathbb{R})$,

† A proof is given by slightly modifying the proof of theorem 4.4 (p 340) of [19].

and $V_i^+ = \cup_{l>0} S_i^+(l)$, where $i=1,2$, are $GL^+(3, \mathbb{R})$ orbits in $\mathbb{R}^6$. On V_1^+ and V_2^+, the measure $\mathrm{d}V = \mathrm{d}x_1\, \mathrm{d}x_2 \ldots \mathrm{d}x_6$ takes the form

$$\mathrm{d}v_1^+(Q) = l^5|\lambda_1-\lambda_2|\,|\lambda_2-\lambda_3|\,|\lambda_3-\lambda_1|\; \mathrm{d}l\, \mathrm{d}\varepsilon_0\, \mathrm{d}\varepsilon_2 \sin\theta\, \mathrm{d}\theta\, \mathrm{d}\phi\, \mathrm{d}\psi$$

$$\mathrm{d}v_2^+(Q) = l^5|\lambda_1-\lambda_2|(\lambda_2+\lambda_3)(\lambda_3+\lambda_1)\; \mathrm{d}l\, \mathrm{d}\varepsilon_0\, \mathrm{d}\varepsilon_2 \sin\theta\, \mathrm{d}\theta\, \mathrm{d}\phi\, \mathrm{d}\psi$$

respectively. These measures are invariant under the action of $SL(3, \mathbb{R})$. This is because, if $Q' = aQa^{\mathrm{T}}$ for $a \in SL(3, \mathbb{R})$ then $\partial(Q'_{11}, \ldots, Q'_{33})/\partial(Q_{11}, \ldots, Q_{33}) = (\det a)^4 = 1$. Since $l^3 = \det Q$ and $\det Q$ is invariant under the action of $SL(3, \mathbb{R})$, $\mathrm{d}\mu_i(Q) = \mathrm{d}v_i^+(Q)/\mathrm{d}l|_{l=1}$, $i=1,2$, are invariant measures on S_1^+ and S_2^+. They are of the form

$$\mathrm{d}\mu_1(Q) = |\lambda_1-\lambda_2|\,|\lambda_2-\lambda_3|\,|\lambda_3-\lambda_1|\; \mathrm{d}\varepsilon_0\, \mathrm{d}\varepsilon_2 \sin\theta\, \mathrm{d}\theta\, \mathrm{d}\phi\, \mathrm{d}\psi \tag{5.5}$$

$$\mathrm{d}\mu_2(Q) = |\lambda_1-\lambda_2|(\lambda_2+\lambda_3)(\lambda_3+\lambda_1)\; \mathrm{d}\varepsilon_0\, \mathrm{d}\varepsilon_2 \sin\theta\, \mathrm{d}\theta\, \mathrm{d}\phi\, \mathrm{d}\psi. \tag{5.6}$$

6. Induced unitary representations of CM(3)

The method of constructing irreducible unitary representations of such a semidirect product group as $CM(3)$ is well known [26]. Let S be one of the $SL(3, \mathbb{R})$ orbits whose origin is O, and K is the isotropy subgroup. Then S is diffeomorphic [27] to $SL(3, \mathbb{R})/K$. Let H^L be the carrier space of an irreducible unitary representation of K labelled by L and $\{|\tau\rangle\}$ be an orthonormal basis for H^L. Let $L_k: H^L \to H^L$ be the representation of $k \in K$. Then

$$L_k|\tau\rangle = \sum_{\tau'} \mathscr{D}_{\tau'\tau}(k)|\tau'\rangle \tag{6.1}$$

where $(\mathscr{D}_{\tau'\tau}(k))$ is the representation matrix.

For $Q \in S$, choose an element $g_Q \in SL(3, \mathbb{R})$ such that $Q = g_Q \cdot O$. The way of choosing g_Q is not unique. However, it suffices that the collection $B_S = \{g_Q |\, Q \in S\}$ becomes a Borel set [28, 29] in $SL(3, \mathbb{R})$. Since $SL(3, \mathbb{R}) \in \mathbb{R}^9$ is locally compact (cf [19], p 186, corollary 8.3), it is equipped with the σ-ring consisting of Borel sets, and for each orbit S we can concretely choose g_Q so that B_S becomes a Borel set. The map $Q \to g_Q$ is, in general, neither differentiable nor continuous. Given $g \in SL(3, \mathbb{R})$, let $Q = g \cdot O$ and $g_Q \in B_S$ be the element such that $Q = g_Q \cdot O$, then $k_g = g_Q^{-1} g$ belongs to K. The decomposition $g = g_Q k_g$ is called the Mackey decomposition [29]. We denote the map $g \to k_g$ by σ. Since the map $Q \to g_Q$ is not continuous, neither is the map $\sigma: g \to k_g$.

Let μ be the invariant or a quasi-invariant measure on S. If we choose $c = \sqrt{\mathrm{d}\mu(g^{-1}Q')/\mathrm{d}\mu(Q')}$ in equation (3.4), then it holds that the completeness relation

$$1 = \int_S |Q'\rangle\, \mathrm{d}\mu(Q')\langle Q'| = \int_S |Q''\rangle\, \mathrm{d}\mu(Q'')\langle Q''|.$$

Let $\mathscr{L}^2(S, \mu, H^L)$ be the set consisting of square integrable functions $f: S \to H^L$. That is, if $f \in \mathscr{L}^2(S, \mu, H^L)$,

$$f(Q) = \sum_\tau f_\tau(Q)|\tau\rangle$$

$$\langle f|f\rangle = \int_S \sum_\tau |f_\tau(Q)|^2\, \mathrm{d}\mu(Q) < +\infty.$$

For $Q \in S$ let $Q = g_Q \cdot O$, where $g_Q \in B_S$. Given $g_0 \in SL(3, \mathbb{R})$ let $Q' = g_0^{-1} \cdot Q$, $g_1 = g_0^{-1} g_Q$ and $g_1 = g_Q k_{g_1}$ be the Mackey decomposition of g_1. Now, for $U_{g_0} \in SL(3, \mathbb{R})$, define a linear map $\rho(U_{g_0}) : \mathscr{L}^2(S, \mu, H^L) \to \mathscr{L}^2(S, \mu, H^L)$ by

$$
\begin{aligned}
(\rho(U_{g_0}) f)(Q) &= L_{k_{g_1}}^{-1} \left(\sum_\tau \langle Q | U_{g_0} | f_\tau \rangle \right) | \tau \rangle \\
&= \sqrt{\frac{\mathrm{d}\mu(g_0^{-1} \cdot Q)}{\mathrm{d}\mu(Q)}}\, L_{k_{g_1}}^{-1} \left(\sum_\tau f_\tau (g_0^{-1} \cdot Q) | \tau \rangle \right) \\
&= \sqrt{\frac{\mathrm{d}\mu(g_0^{-1} \cdot Q)}{\mathrm{d}\mu(Q)}} \sum_{\tau, \tau'} \mathscr{D}(k_{g_1})_{\tau\tau'} f_{\tau'} (g_0^{-1} \cdot Q) | \tau \rangle .
\end{aligned}
\tag{6.2}
$$

Then $\rho(U_{g_0})$ is a strongly continuous unitary representation of U_{g_0}. Note that $g \to k_g$ is not continuous and therefore $g \to \mathscr{D}_{\tau\tau'}(k_g)$ is not continuous, but $U_{g_0} \to \rho(U_{g_0})$ is continuous. That is, even if $g \to \mathscr{D}_{\tau\tau'}(k_g)$ is not continuous, if $\|g_0 - \tilde{g}_0\|$ is sufficiently small $|\mathscr{D}_{\tau\tau'}(k_{g_1}) - \mathscr{D}_{\tau\tau'}(k_{\tilde{g}_1})|$ becomes arbitrary small for $k_{g_1} = \sigma(g_0^{-1} g_Q)$ and $k_{\tilde{g}_1} = \sigma(\tilde{g}_0^{-1} g_Q)$. Finally, for $U_\kappa U_g \in CM(3)$ if we define

$$\rho(U_\kappa U_g) : \mathscr{L}^2(S, \mu, H^L) \to \mathscr{L}^2(S, \mu, H^L)$$

by

$$(\rho(U_\kappa U_g) f)(Q) = \chi_Q(U_\kappa)(\rho(U_g) f)(Q)$$

then $U_\kappa U_g \to \rho(U_\kappa U_g)$ is an irreducible unitary representation of $CM(3)$.

If L is the trivial representation of K, then equation (6.2) becomes

$$(\rho(U_{g_0}) f)(Q) = \sqrt{\frac{\mathrm{d}\mu(g_0^{-1} \cdot Q)}{\mathrm{d}\mu(Q)}}\, f(g_0^{-1} \cdot Q) \tag{6.3}$$

and $\mathscr{L}^2(\mu, S, H^L)$ becomes $\mathscr{L}^2(S, \mu) = \{ f : S \to \mathbb{C} \,|\, \int_S |f(Q)|^2 \,\mathrm{d}\mu(Q) < +\infty \}$. As $\int_S \Sigma_\tau |f_\tau(Q)|^2 \,\mathrm{d}\mu(Q) < +\infty$, obviously $f_\tau \in \mathscr{L}^2(S, \mu)$. Therefore, if $\{u_n(Q)\}$ is a basis for $\mathscr{L}^2(S, \mu)$, then $\{u_n(Q) \otimes | \tau \rangle\}$ serves as a basis for $\mathscr{L}^2(S, \mu, H^L)$.

7. Basis for $\mathscr{L}^2(S, \mu)$

Let us introduce some basis for $\mathscr{L}^2(S, \mu)$ by considering the structure of S. First we note that if $f : X \to Y$ is a homeomorphism, μ_X and μ_Y are quasi-invariant measures on X and Y, respectively and $\{\phi_n(x)\}$ is a basis for $\mathscr{L}^2(X, \mu_X)$, then the collection of $\psi_n(y)$ defined by

$$\psi_n(y) = \sqrt{\frac{\mathrm{d}\mu_X(x)}{\mathrm{d}\mu_Y(y)}}\, (\phi \cdot f^{-1})(y) = \sqrt{\frac{\mathrm{d}\mu_X(x)}{\mathrm{d}\mu_Y(y)}}\, \phi_n(x)$$

$$y = f(x) \in Y \qquad x = f^{-1}(y) \in X$$

serves as a basis for $\mathscr{L}^2(Y, \mu_Y)$. Also, we shall make use of the fact that if $g : X \to Y$ is a surjection and $h : X \to \mathbb{C}$ is constant on the inverse image $g^{-1}(y)$ of $y \in Y$, then h can be regarded as a function on Y.

7.1. *Basis for $\mathscr{L}2(S_5^+, \mu_5)$*

If $Q\in S_5^+$, then $Q=v\tilde{Q}$, where $v\in\mathbb{R}^+$ and $\tilde{Q}\in M_5\cong P^2$. The map $(v, \tilde{Q})\to v\tilde{Q}\in S_5^+$ is continuous. Also, the inverse of the map is continuous. Because, for $Q\in S_5^+$, $v=\operatorname{Tr} Q$ and $\tilde{Q}=Q/v$ are continuous. Thus $S_5^+\cong\mathbb{R}^+\times P^2$. Let $\mathrm{d}\mu_{R^+}(v)=\mathrm{e}^{-v}\,\mathrm{d}v$. Then Laguerre polynomials [30] $L_n(v)=\mathrm{e}^v\,\mathrm{d}^n(\mathrm{e}^{-v}\,v^n)/\mathrm{d}v^n$, satisfying the orthogonality

$$\int_0^\infty L_n(v)L_{n'}\,\mathrm{d}\mu_{\mathbb{R}^+}(v)=(n!)^2\delta_{nn'}$$

are complete in $\mathscr{L}^2(\mathbb{R}^+, \mu_{\mathbb{R}^+})$. Spherical harmonics $Y_{lm}(\theta, \phi)$ are complete in $\mathscr{L}^2(S^2, \mu_{S^2})$, where $\mathrm{d}\mu_{S^2}(\theta, \phi)=\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi$. Functions on S^2 which take the same value on a pair of antipodal points can be regarded as functions on P^2 (cf equation (4.1)). As $Y_{lm}(\pi-\theta, \phi+\pi)=(-1)^l Y_{lm}(\theta, \phi)$, the set $\{Y_{lm}\,|\,l \text{ even}\}$ is complete in $\mathscr{L}^2(P^2, \mu_{S^2})$. The difference of the measures $\mathrm{d}\mu_5(Q)$ and $\mathrm{d}\mu_{\mathbb{R}^+}(v)\,\mathrm{d}\mu_{S^2}(\theta, \phi)$ is compensated by merely multiplying $L_n(v)\,Y_{lm}(\theta, \phi)$ by $\mathrm{e}^{-v/2}/v$. Thus,

$$\{(\mathrm{e}^{-v/2}/v)L_n(v)\,Y_{lm}(\theta, \phi)|n\in\mathbb{N}, l \text{ even}\}$$

is an orthogonal basis for $\mathscr{L}^2(S_5^+, \mu_5)$.

7.2. *Basis for $\mathscr{L}^2(S_3^+, \mu_3)$*

If $Q\in S_3^+$, then $Q=r\operatorname{diag}(\lambda_1, \lambda_2, 0)r^{\mathrm{T}}$, where $r\in SO(3)$ and $\lambda_1, \lambda_2\in\mathbb{R}^+$. Let $v=\sqrt{\lambda_1\lambda_2}$ and $\mathrm{e}^\varepsilon=\sqrt{\lambda_1/\lambda_2}$, then $v\in\mathbb{R}^+$, $\varepsilon\in\mathbb{R}$ and $Q=vr\operatorname{diag}(\mathrm{e}^\varepsilon, \mathrm{e}^{-\varepsilon}, 0)r^{\mathrm{T}}$. The orbit S_3^+ is the Cartesian product of $\mathbb{R}^+$ and $\tilde{S}_3=\{r\operatorname{diag}(\mathrm{e}^\varepsilon, \mathrm{e}^{-\varepsilon}, 0)r^{\mathrm{T}}|\,\varepsilon\in\mathbb{R}, r\in SO(3)\}$. In fact, if $Q\in S_3^+$, then $v=\sqrt{[(\operatorname{Tr} Q)^2-\operatorname{Tr} Q^2]/2}$ and $\tilde{Q}=Q/v$ are continuous, and the inverse $(v, \tilde{Q})\to v\tilde{Q}$ is continuous. Let

$$SO(2)=\left\{h(\phi)=\begin{pmatrix}\cos\phi & -\sin\phi & 0\\ \sin\phi & \cos\phi & 0\\ 0 & 0 & 1\end{pmatrix}\middle|\,\phi\in\mathbb{R}\right\}.$$

Since

$$SO(3)=\{h(\phi)g(\theta)h(\psi)|h(\phi), h(\psi)\in SO(2), \theta\in[0, \pi]\}$$

and $\{\operatorname{diag}(\mathrm{e}^\varepsilon, \mathrm{e}^{-\varepsilon}, 0)|\,\varepsilon\in\mathbb{R}\}\cong\mathbb{R}$, $\tilde{S}_3$ is the image set of the map

$$\tilde{f}_3: SO(2)\times[0, \pi]\times SO(2)\times\mathbb{R}\to\mathbb{R}^6$$

defined by

$$\tilde{f}_3(h(\phi), \theta, h(\psi), \varepsilon)=h(\phi)g(\theta)h(\psi)\operatorname{diag}(\mathrm{e}^\varepsilon, \mathrm{e}^{-\varepsilon}, 0)h^{\mathrm{T}}(\psi)g^{\mathrm{T}}(\theta)h^{\mathrm{T}}(\phi).$$

However, there exists a more convenient parametrization of $Q\in\tilde{S}_3$. Let $\tilde{X}_3=\{h\operatorname{diag}(\mathrm{e}^\varepsilon, \mathrm{e}^{-\varepsilon}, 0)h^{\mathrm{T}}|h\in SO(2)\}\subset\tilde{S}_3$. Define a map $\tilde{g}_3: SO(2)\times[0, \pi]\times\tilde{X}_3\to\mathbb{R}^6$ by $\tilde{g}_3(h(\phi), \theta, P)=h(\phi)g(\theta)Pg^{\mathrm{T}}(\theta)h^{\mathrm{T}}(\phi)$, where $P=h(\psi)\operatorname{diag}(\mathrm{e}^\varepsilon, \mathrm{e}^{-\varepsilon}, 0)h^{\mathrm{T}}(\psi)\in\tilde{X}_3$, then $\tilde{f}_3(h(\phi), \theta, h(\psi), \varepsilon)=\tilde{g}_3(h(\phi), \theta, P)$ and $\tilde{S}_3=\tilde{g}_3(SO(2)\times[0, \pi]\times\tilde{X}_3)$. Since

$$\tilde{g}_3(h(\phi), 0, P)=(h(\phi)h(\psi))\operatorname{diag}(\mathrm{e}^\varepsilon, \mathrm{e}^{-\varepsilon}, 0)(h(\phi)h(\psi))^{\mathrm{T}}\in\tilde{X}_3$$

$$\tilde{g}_3(h(\phi), \pi, P)=(h(\phi)h^{-1}(\psi))\operatorname{diag}(\mathrm{e}^\varepsilon, \mathrm{e}^{-\varepsilon}, 0)(h(\phi)h^{-1}(\psi))^{\mathrm{T}}\in\tilde{X}_3$$

$SO(2)\times\{0\}\times\tilde{X}_3$ and $SO(2)\times\{\pi\}\times\tilde{X}_3$ are projected on to $\tilde{X}_3$. If $\theta\in(0, \pi)$, then

$\tilde{g}_3(h(\phi), \theta, P)$ does not belong to $\tilde{X}_3$. We shall consider functions from $SO(2)\times[0,\pi]\times\tilde{X}_3$ to $\mathbb{C}$, which are constant on $\tilde{g}_3^{-1}(Q)$.

First we show that $\tilde{X}_3\cong\mathbb{R}^2$. Let p_{ij} denote the (i,j) component of $P=h(\psi)\,\mathrm{diag}(e^{\varepsilon}, e^{-\varepsilon}, 0)h^{\mathrm{T}}(\psi)\in\tilde{X}_3$. Then

$$u=p_{11}+p_{22}=(e^{\varepsilon}+e^{-\varepsilon}) \qquad v=p_{11}-p_{22}=(e^{\varepsilon}-e^{-\varepsilon})\cos 2\psi$$

$$w=2p_{12}=(e^{\varepsilon}-e^{-\varepsilon})\sin 2\psi \tag{7.1}$$

and the other components of P are 0. Since $u^2-(v^2+w^2)=4$ and $u>0$, $\tilde{X}_3$ is a sheet of the hyperboloid of two sheets. The projection $(u,v,w)\to(v,w)$ and its inverse $(v,w)\to(\sqrt{4+v^2+w^2}, v, w)$ are differentiable, and consequently $\tilde{X}_3\cong\mathbb{R}^2$.

Let $H_n(u)$ denote a Hermite polynomial [31]. Then

$$\{\exp[-(v^2+w^2)/2]H_m(v)H_n(w)|n, m=0,1,2,\ldots\}$$

is complete in $\mathscr{L}^2(\mathbb{R}^2, \mu_{\mathbb{R}^2})$, where $d\mu_{\mathbb{R}^2}(v,w)=dv\,dw$. However, it is more convenient to adopt their linear combinations

$$\begin{aligned}\sum c_{mn}H_m(v)H_n(w)&=\exp[-(v^2+w^2)/2]L_j^{|k|}(v^2+w^2)(v+iw)^k &&\text{if } k\geqslant 0\\ &=\exp[-(v^2+w^2)/2]L_j^{|k|}(v^2+w^2)(v-iw)^{-k} &&\text{if } k<0\end{aligned}$$

where $L_j^{|k|}$ is the Laguerre polynomial [30], k an integer and j a non-negative integer. Substituting $\rho\cos 2\psi$ and $\rho\sin 2\psi$, where $\rho=(e^{\varepsilon}-e^{-\varepsilon})$, for v and w respectively, we have basis functions

$$\phi_{j,k}(\rho,\psi)=e^{-\rho^2/2}\rho^{|k|}L_j^{|k|}(\rho^2)\,e^{2ik\psi} \tag{7.2}$$

well defined on $\tilde{X}_3$. The function $\phi_{j,k}(\rho,\psi)$ is invariant under the transformation $(\rho,\psi)\to(-\rho,\psi\pm\pi/2)$. This invariance arises from $h(\psi)\,\mathrm{diag}(e^{\varepsilon}, e^{-\varepsilon}, 0)h^{\mathrm{T}}(\psi)=h(\psi\pm\pi/2)\,\mathrm{diag}(e^{-\varepsilon}, e^{\varepsilon}, 0)h^{\mathrm{T}}(\psi\pm\pi/2)$. In this parametrization of (v,w) by (ρ,ψ) any function on $\tilde{X}_3$ must satisfy the invariance. Therefore,

$$\begin{aligned}\int_{-\infty}^{0}\rho\,d\rho\int_0^{2\pi}d\psi\,F(\rho,\psi)&=\int_0^{\infty}\rho\,d\rho\int_{\pm\pi/2}^{2\pi\pm\pi/2}d\psi\,F(\rho,\psi)\\ &=\int_0^{\infty}\rho\,d\rho\int_0^{2\pi}d\psi\,F(\rho,\psi)\end{aligned}$$

and consequently, in equation (7.2), we may restrict the domain of ρ to $[0,-\infty)$.

For $h(\phi)\in SO(2)$, let $z_1=e^{i\phi}$. Then the set $\{z_1^M\,d^J_{MK}(\theta)|J=0,1,\ldots,M=-J,\ldots,J\}$, where K is arbitrarily chosen for each J, is complete in the space of square integrable functions on $SO(2)\times[0,\pi]$. Therefore, $\{\phi_{j,k}(\rho,\psi)z_1^M\,d^J_{MK}(\theta)\}$ is complete in the space of square integrable functions on $SO(2)\times[0,\pi]\times\tilde{X}_3$.

Now, if $\theta\in(0,\pi)$ and $Q=\tilde{g}_3(h(\phi),\theta,P)$ then $\tilde{g}_3^{-1}(Q)$ consists of two points $(h(\phi),\theta,P)$ and $(h(\phi+\pi),\pi-\theta,g(\pi)Pg(\pi))$. Since $g(\pi)h\,\mathrm{diag}(e^{\varepsilon}, e^{-\varepsilon}, 0)h^{\mathrm{T}}g(\pi)=h^{\mathrm{T}}\,\mathrm{diag}(e^{\varepsilon}, e^{-\varepsilon}, 0)h$, if the value of function $\phi_{j,k}$ at P is $\phi_{j,k}(\rho,\psi)$, its value at $g(\pi)Pg(\pi)$ is $\phi_{j,k}(\rho,-\psi)$. Also, $e^{iM(\phi+\pi)}\,d^J_{MK}(\pi-\theta)=(-1)^J e^{iM\phi}\,d^J_{M,-K}(\theta)$. Therefore,

$$\psi(Q)=e^{-\rho^2/2}\rho^{|k|}L_j^{|k|}(\rho^2)\,e^{iM\phi}(d^J_{MK}(\theta)\,e^{2ik\psi}+(-1)^J d^J_{M,-K}(\theta)\,e^{-2ik\psi}) \tag{7.3}$$

is well defined on $\tilde{S}_3 - \tilde{X}_3$. We shall extend the domain of this function to $\tilde{S}_3$. For $Q \in \tilde{S}_3 - \tilde{X}_3$, $\lim_{\theta\to 0,\pi} Q$ belongs to $\tilde{X}_3$. On the other hand,

$$\lim_{\theta\to 0} \psi(Q) = \mathrm{e}^{-\rho^2/2}\, \rho^{|k|} L_j^{|k|}(\rho^2)(\delta_{MK}\, \mathrm{e}^{\mathrm{i}K\phi}\, \mathrm{e}^{2\mathrm{i}k\psi} + (-1)^J \delta_{M,-K}\, \mathrm{e}^{-\mathrm{i}K\phi}\, \mathrm{e}^{-2\mathrm{i}k\psi}).$$

The limit $\lim_{\theta\to 0} \psi(Q)$ becomes a function on $\tilde{X}_3$ if and only if $K=2k$. This is because, if and only if $K=2k$, $\lim_{\theta\to 0} \psi(Q)$ is a function of $h(\phi)h(\psi) \in SO(2)$ and is a linear combination of the functions given in equation (7.2). Also, if $K=2k$,

$$\lim_{\theta\to\pi} \psi(Q) = \mathrm{e}^{-\rho^2/2} \rho^{|k|} L_j^{|k|}(\rho^2)((-1)^J \delta_{M,-K}\, \mathrm{e}^{-\mathrm{i}2k(\phi-\psi)} + \delta_{MK}\, \mathrm{e}^{\mathrm{i}2k(\phi-\psi)})$$

which is a function of $h(\phi)h^{-1}(\psi) \in SO(2)$, and well defined on $\tilde{X}_3$.

Let $\gamma = \{n, j, J, M, 2k\}$, $\Gamma = \{\gamma\}$ and

$$\psi_\gamma(Q) = L_n(v)\, \mathrm{e}^{-\rho^2/2} \rho^{|k|} L_j^{|k|}(\rho^2)(D^J_{M,2k}(\phi, \theta, \psi) + (-1)^J D^J_{M,-2k}(\phi, \theta, \psi)). \tag{7.4}$$

In the variables ρ and v,

$$\mathrm{d}\mu_3(Q) = 2v^4\, \mathrm{d}v \frac{\rho\, \mathrm{d}\rho}{\sqrt{\rho^2+4}} \sin\theta\, \mathrm{d}\theta\, \mathrm{d}\phi\, \mathrm{d}\psi.$$

Therefore, $\{\sqrt[4]{\rho^2+4}\, v^{-2}\, \mathrm{e}^{-v/2}\, \psi_\gamma(Q) | \gamma \in \Gamma\}$ is a basis for $\mathscr{L}^2(S_3^+, \mu_3)$.

7.3. Basis for $\mathscr{L}^2(S_4, \mu_4)$

A basis for $\mathscr{L}^2(S_4, \mu_4)$ can be identified in the same way as above. If $Q \in S_4$, then $Q = vr\, \mathrm{diag}(\mathrm{e}^\varepsilon - \mathrm{e}^{-\varepsilon}, 0) r^{\mathrm{T}}$, where $v \in \mathbb{R}^+$, $\varepsilon \in \mathbb{R}$ and $r \in SO(3)$. Therefore, S_4 is a Cartesian product of $\mathbb{R}^+$ and $\tilde{S}_4 = \{r\, \mathrm{diag}(\mathrm{e}^\varepsilon, -\mathrm{e}^{-\varepsilon}, 0) r^{\mathrm{T}} | r \in SO(3),\ \varepsilon \in \mathbb{R}\}$. Let $\tilde{X}_4 = \{h\, \mathrm{diag}(\mathrm{e}^\varepsilon, -\mathrm{e}^{-\varepsilon}, 0) h^{\mathrm{T}} | h \in SO(2), \varepsilon \in \mathbb{R}\}$. If $P \in \tilde{X}_4$, then

$$u = p_{11} + p_{22} = (\mathrm{e}^\varepsilon - \mathrm{e}^{-\varepsilon}) \qquad v = p_{11} - p_{22} = (\mathrm{e}^\varepsilon + \mathrm{e}^{-\varepsilon}) \cos 2\psi$$

$$w = 2p_{12} = (\mathrm{e}^\varepsilon + \mathrm{e}^{-\varepsilon}) \sin 2\psi \tag{7.5}$$

and the other components of P are 0. Since $v^2 + w^2 - u^2 = 4$, $\tilde{X}_4$ is the hyperboloid of one sheet, which is diffeomorphic to $\mathbb{R} \times S^1$, $S^1 = \{(n_1, n_2) | n_1^2 + n_2^2 = 1\}$. This is because both $(u, v, w) \to (u, v/\sqrt{v^2+w^2}, w/\sqrt{v^2+w^2}) \in \mathbb{R} \times S^1$ and $(u, n_1, n_2) \to (u, \sqrt{u^2+4}n_1, \sqrt{u^2+4}n_2) \in \tilde{S}_4$ are differentiable. Therefore, the collection of functions $\mathrm{e}^{-u^2/2} H_m(u)\, \mathrm{e}^{2\mathrm{i}k\psi}$ serves as a basis for the space of square integrable functions on $\tilde{X}_4$. If we make

$$\psi_\gamma(Q) = L_n(v)\, \mathrm{e}^{-u^2/2} H_m(u)(D^J_{M,2k}(\phi, \theta, \psi) + (-1)^J D^J_{M,-2k}(\phi, \theta, \psi))$$

where $\gamma = \{n, m, J, M, 2k\}$, then the collection $\{v^{-2}\, \mathrm{e}^{-v/2}\, \psi_\gamma(Q) | \gamma \in \Gamma\}$ becomes a basis for $\mathscr{L}^2(S_4, \mu_4)$.

7.4. Basis for $\mathscr{L}^2(S_2^+, \mu_2)$

We shall show that S_2^+ is homeomorphic to S_3^+ and infer that the collection of the functions of the form (7.4) also serves as a basis for $\mathscr{L}^2(S_2^+, \mu_2)$. Let $f_2: SO(2) \times [0, \pi] \times SO(2) \times \mathbb{R}^{+2} \to S_2^+$ and $f_3: SO(2) \times [0, \pi] \times SO(2) \times \mathbb{R}^{+2} \to S_3^+$ be defined by

$$f_2(h(\phi), \theta, h(\psi), \lambda_1, \lambda_2) = h(\phi)g(\theta)h(\psi)\,\mathrm{diag}(-\lambda_1, -\lambda_2, 1/\lambda_1\lambda_2)h^{\mathrm{T}}(\psi)g^{\mathrm{T}}(\theta)h^{\mathrm{T}}(\phi),$$

$$f_3(h(\phi), \theta, h(\psi), \lambda_1, \lambda_2) = h(\phi)g(\theta)h(\psi)\,\mathrm{diag}(\lambda_1, \lambda_2, 0)h^{\mathrm{T}}(\psi)g^{\mathrm{T}}(\theta)h^{\mathrm{T}}(\phi).$$

Let

$$X_2 = \{h\,\mathrm{diag}(-\lambda_1, -\lambda_2, 1/\lambda_1\lambda_2)h^{\mathrm{T}} | h \in SO(2)\}$$

$$X_3 = \{h\,\mathrm{diag}(\lambda_1, \lambda_2, 0)h^{\mathrm{T}} | h \in SO(2)\}.$$

Then if and only if $\theta = 0$ or $\pi, f_2(h(\phi), \theta, h(\psi), \lambda_1, \lambda_2) \in X_2$ and $f_3(h(\phi), \theta, h(\psi), \lambda_1, \lambda_2) \in X_3$. Also, if $\theta = 0$ or π, then f_2 and f_3 depend on $h(\phi), h(\psi) \in SO(2)$ through the product $h(\phi)h(\psi)$ or $h(\phi)h^{-1}(\psi)$.

If

$$P_3 = h(\psi)\,\mathrm{diag}(\lambda_1, \lambda_2, 0)h^{\mathrm{T}}(\psi) \in X_3$$

and

$$P_2 = h(\psi)\,\mathrm{diag}(-\lambda_1, -\lambda_2, 1/\lambda_1\lambda_2)h^{\mathrm{T}}(\psi) \in X_2$$

then

$$P_3 = \begin{pmatrix} p_{11} & p_{12} & 0 \\ p_{12} & p_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad P_2 = \begin{pmatrix} -p_{11} & -p_{12} & 0 \\ -p_{12} & -p_{22} & 0 \\ 0 & 0 & 1/(p_{11}p_{22} - p_{12}^2) \end{pmatrix}.$$

Since the map $P_2 \to P_3$ and its inverse are continuous, $X_2 \cong X_3$. Let η_1 denote the homeomorphism from X_2 to X_3. Obviously

$$\eta_2 : SO(2) \times [0, \pi] \times X_2 \to SO(2) \times [0, \pi] \times X_3$$

defined by

$$\eta_2(h(\phi), \theta, P_2) = (h(\phi), \theta, \eta_1(P_2))$$

is a homeomorphism.

Now, if we define $g_2 : SO(2) \times [0, \pi] \times X_2 \to S_2^+$ and $g_3 : SO(2) \times [0, \pi] \times X_3 \to S_3^+$ by $g_2(h(\phi), \theta, P_2) = h(\phi)g(\theta)P_2 g^{\mathrm{T}}(\theta)h^{\mathrm{T}}(\phi)$ and $g_3(h(\phi), \theta, P_2) = h(\phi)g(\theta)P_3 g^{\mathrm{T}}(\theta)h^{\mathrm{T}}(\phi)$, then both of $\zeta = g_3 \cdot \eta_2 \cdot g_2^{-1}$ and $\bar{\zeta} = g_2 \cdot \eta_2^{-1} \cdot \eta_2^{-1} \cdot g_3^{-1}$ become well defined continuous maps whose compositions $\zeta \cdot \bar{\zeta}$ and $\bar{\zeta} \cdot \zeta$ are identities. Therefore $S_2^+ \cong S_3^+$. If we adopt the variables $\nu = \sqrt{\lambda_1\lambda_2}$ and $\rho = \sqrt{\lambda_1/\lambda_2} - \sqrt{\lambda_2/\lambda_1}$ instead of ε_0 and ε_2, the invariant measure on S_2^+ becomes

$$\mathrm{d}\mu_2(Q) = \frac{\sqrt{3}}{2}\left(\frac{\nu^2 + \nu^{-4} + \sqrt{\rho^2 + 4}}{\sqrt{\rho^2 + 4}}\right)\mathrm{d}\nu\,\rho\,\mathrm{d}\rho\,\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi\,\mathrm{d}\psi.$$

Therefore, if $\psi_\gamma(Q)$ is given by the right side of equation (7.4), the collection $\{\mathrm{e}^{-\nu/2}(\sqrt[4]{\rho^2 + 4}/\sqrt{\nu^2 + \nu^{-4} + \sqrt{\rho^2 + 4}})\psi_\gamma(Q) | \gamma \in \Gamma\}$ is an orthogonal basis for $\mathscr{L}^2(S_2^+, \mu_2)$.

7.5. Basis for $\mathscr{L}^2(S_1^+, \mu_1)$

Finally we shall identify a basis for $\mathscr{L}^2(S_1^+, \mu_1)$. Let $s = \{A \in sl(3, \mathbb{R}) | A = A^T\}$ be the subspace of $sl(3, \mathbb{R}) \cong \mathbb{R}^8$. Let Ξ be the two-dimensional subspace of $s \cong \mathbb{R}^5$ consisting of diagonal matrices. Then $\exp: \Xi \to \Lambda$ is a diffeomorphism. If $Q \in S_1^+$, then $Q = r\lambda^2 r^T$, where $\lambda \in \Lambda$ and $r \in SO(3)$. Since $\lambda = \exp(\xi)$ for a unique $\xi \in \Xi$, $Q = r\exp(\xi) r^T = \exp(2A)$, where $A = r\xi r^T \in s$. Any $A \in s$ can be represented in the form $r\xi r^T$ for some $\xi \in \Xi$ and $r \in SO(3)$. The map $\exp: s \to S_1^+$ defined by the matrix power series is a diffeomorphism [32]. Therefore, any basis for $\mathscr{L}^2(\mathbb{R}^5, \mu_{\mathbb{R}^5})$ becomes a basis for $\mathscr{L}^2(S_1^+, \mu_1)$, if the difference of the invariant measures is suitably adjusted. However, in order to show the relation of the $CM(3)$ model and the Bohr model, we will explain some detail.

Let

$$E_0 = \begin{pmatrix} -1/\sqrt{6} & 0 & 0 \\ 0 & -1/\sqrt{6} & 0 \\ 0 & 0 & \sqrt{2/3} \end{pmatrix} \qquad E_{\pm 2} = \begin{pmatrix} \frac{1}{2} & \mp i/2 & 0 \\ \mp i/2 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$E_{\pm 1} = \begin{pmatrix} 0 & 0 & \mp\frac{1}{2} \\ 0 & 0 & i/2 \\ \mp\frac{1}{2} & i/2 & 0 \end{pmatrix}.$$

Then, these matrices form an orthonormal basis for s with respect to the scalar product $(E_\mu, E_\nu) = \mathrm{Tr}\, E_\mu \bar{E}_\nu$. If $Q = (Q_{ij})$ is a 3×3 real symmetric matrix, then

$$Q = Q^0 I + \sum_{\mu=-2}^{2} Q_\mu^2 E_\mu \tag{7.6}$$

where Q^0 and Q_μ^2s are expressions given by equations (1.2) and (1.3). The matrices E_μ transform under the action of $r = h(\phi)g(\theta)h(\psi) \in SO(3)$, as

$$rE_\mu r^T = \sum_{\nu=-2}^{2} D_{\nu\mu}^2(\phi, \theta, \psi) E_\nu. \tag{7.7}$$

Now, if $A \in s$ then $A = r\xi r^T$, for some $\xi \in \Xi$ and $r \in SO(3)$. Since $\{E_0, (E_2 + E_{-2})/\sqrt{2}\}$ is a basis for Ξ, $\xi = \varepsilon_0 E_0 + \varepsilon_2 (E_2 + E_{-2})/\sqrt{2}$ for some $\varepsilon_0, \varepsilon_2 \in \mathbb{R}$. Therefore, any $A \in s$ is the linear combination $\sum_{\mu=-2}^{2} \alpha_\mu E_\mu$, whose coefficients are

$$\alpha_\mu = \varepsilon_0 D_{\mu,0}^2(\phi, \theta, \psi) + \frac{\varepsilon_2}{\sqrt{2}} [D_{\mu,2}^2(\phi, \theta, \psi) + D_{\mu,-2}^2(\phi, \theta, \psi)] \qquad \mu = 0, \pm 1, \pm 2. \tag{7.8}$$

This expression is essentially the same as expression (1.11). Thus, $\mathbb{R}^5 = \{\alpha_\mu\}$ of the Bohr model is the tangent space of S_1^+ at the origin.

The Euclidean measure $\Pi_{\nu=-2}^{2}\, \mathrm{d}\alpha_\nu$ on s is, apart from the numeral factors,

$$\mathrm{d}\mu_s(A) = \beta^3 \left| \prod_{k=1}^{3} \sin\left(\gamma - \frac{2\pi}{3} k\right) \right| \beta\, \mathrm{d}\beta\, \mathrm{d}\gamma \sin\theta\, \mathrm{d}\theta\, \mathrm{d}\phi\, \mathrm{d}\psi. \tag{7.9}$$

On the other hand, if we substitute $\beta\cos\gamma$ and $\beta\sin\gamma$ for ε_0 and ε_2 in equation (5.5), respectively, we have

$$\mathrm{d}\mu_1(Q)=\left|8\prod_{k=1}^{3}\sinh\left[\sqrt{2}\beta\sin\left(\gamma-\frac{2\pi}{3}k\right)\right]\right|\beta\,\mathrm{d}\beta\,\mathrm{d}\gamma\sin\theta\,\mathrm{d}\phi\,\mathrm{d}\psi. \tag{7.10}$$

Let $\psi_\omega(\beta,\gamma,\phi,\theta,\psi)$ denote the eigenfunction [33–35] of the five-dimensional harmonic oscillator, where ω denotes a set of quantum numbers. Let $\Omega=\{\omega\}$. Then $\{\psi_\omega(\beta,\gamma,\phi,\theta,\psi)|\omega\in\Omega\}$ is a basis for $\mathscr{L}^2(s,\mu_s)$. Therefore,

$$\left\{\sqrt{\left(\left|\prod_{k=1}^{3}\frac{\beta\sin(\gamma-2\pi k/3)}{\sinh[\sqrt{2}\beta\sin(\gamma-2\pi k/3)]}\right|\right)}\psi_\omega(\beta,\gamma,\phi,\theta,\psi)|\omega\in\Omega\right\}$$

serves as a basis for $\mathscr{L}^2(S_1^+,\mu_1)$. As is desired from $\lim_{\beta\to 0}Q=I+2A$, $\lim_{\beta\to 0}\mathrm{d}\mu_1(Q)$ is proportional to $\mathrm{d}\mu_s(A)$.

8. Representation of *cm*(3)

Let S, O, K and k denote an $SL(3,\mathbb{R})$ orbit, its origin, the isotropy subgroup and the Lie algebra of K. Let $\hat{Z}\in cm(3)$ and $U_\varepsilon=\mathrm{e}^{\varepsilon\hat{Z}}$. The map $\hat{Z}\to\sigma(\hat{Z})$ defined by

$$\lim_{\varepsilon\to 0}\frac{(\rho(U_\varepsilon)f)(Q)-f(Q)}{\varepsilon}=\sigma(Z)f(Q) \tag{8.1}$$

is a representation of $cm(3)$. Since

$$\rho(\mathrm{e}^{\mathrm{i}\varepsilon Q_{kl}})f(Q')=\mathrm{e}^{\mathrm{i}\varepsilon Q'_{kl}}f(Q')$$

$\sigma(Q_{ij})=Q'_{ij}$. For $\hat{Z}\in sg$, the left side of equation (8.1) can be calculated with equation (6.2). We denote by $\hat{Z}$ the element of sg which is mapped to $Z\in sl(3,\mathbb{R})$ by the isomorphism $h|sg:sg\to sl(3,\mathbb{R})$.

Let

$$A_1=\begin{pmatrix}0&0&0\\0&0&-1\\0&1&0\end{pmatrix}\qquad A_2=\begin{pmatrix}0&0&1\\0&0&0\\-1&0&0\end{pmatrix}\qquad A_3=\begin{pmatrix}0&-1&0\\1&0&0\\0&0&0\end{pmatrix}.$$

Then $\{A_i,E_\mu|i=1,2,3,\mu=0,\pm1,\pm2\}$ is a basis for $sl(3,\mathbb{R})$. In some step of calculation, it is convenient to use real symmetric matrices

$$B_1=\begin{pmatrix}0&0&0\\0&0&1\\0&1&0\end{pmatrix}\qquad B_2=\begin{pmatrix}0&0&1\\0&0&0\\1&0&0\end{pmatrix}$$

$$B_3=\begin{pmatrix}0&1&0\\1&0&0\\0&0&0\end{pmatrix}\qquad H_2=\frac{1}{\sqrt{2}}\begin{pmatrix}1&0&0\\0&-1&0\\0&0&0\end{pmatrix}$$

instead of $E_{\pm 2}$ and $E_{\pm 1}$. Let $Z_1=(B_1-A_1)$ and $Z_2=(B_2+A_2)$. By means of those matrices, the Lie algebras of isotropy subgroups are

$$so(3)=\text{span}\{A_1, A_2, A_3\} \qquad so(2,1)=\text{span}\{B_1, B_2, A_3\}$$

$$m(2)=\text{span}\{Z_1, Z_2, A_3\} \qquad mh(2)=\text{span}\{Z_1, Z_2, B_3\}$$

$$sa(2)=\text{span}\{Z_1^{\mathrm{T}}, Z_2^{\mathrm{T}}, A_3, B_3, H_2\}.$$

Let $k \to L_k$ be an irreducible unitary representation of K in H^L. We denote by $\sigma^L(Z)$ the skew Hermitian operator which represents $Z \in \boldsymbol{k}$. Then, if $k=\mathrm{e}^{\varepsilon Z} \in K$,

$$L_k=\exp(\varepsilon\sigma^L(Z)). \tag{8.2}$$

In sections 4 and 7, $Q \in S$ is parametrized as $Q=r\lambda Or^{\mathrm{T}}$, where $\lambda \in \Lambda$ and $r \in SO(3)$. The expression is rewritten as $Q=r\lambda^{1/2}O(r\lambda^{1/2})^{\mathrm{T}}$, where $\lambda^{1/2}=\text{diag}(\sqrt{\lambda_2}, \sqrt{\lambda_2}, \sqrt{\lambda_3})$ for $\lambda=\text{diag}(\lambda_1, \lambda_2, \lambda_3) \in \Lambda$. If we denote $r\lambda^{1/2}$ by g_Q, then $Q=g_Q O g_Q^{\mathrm{T}}=g_Q \cdot O$. It is convenient to take $\exp(\varepsilon_0 E_0)$ or $\exp(\varepsilon_0 E_0+\varepsilon_2 H_2)$ as $\lambda^{1/2}$ according to whether S is S_5^+ or the other orbits.

8.1. *Angular momentum operators*

If $g_\varepsilon=\exp(\varepsilon A_k)$, then $U_{g_\varepsilon}=\exp(\varepsilon\hat{A}_k)$. Since $g_\varepsilon^{-1}g_Q=r'\lambda^{1/2}$, where $r' \in SO(3)$, $\sqrt{\mathrm{d}\mu(g_\varepsilon^{-1} \cdot Q)/\mathrm{d}\mu(Q)}=1$. The expressions $\sigma(\hat{A}_k)$ have two different forms according to whether S is S_5^+ or the other orbits.

If $S=S_5^+$, we can choose $h(\phi)g(\theta)$ as $r \in SO(3)$. However,

$$\mathrm{e}^{-\varepsilon A_k}h(\phi)g(\theta)=h(\phi+\delta\phi)g(\theta+\delta\theta)h(\delta\psi). \tag{8.3}$$

As, $h(\delta\psi)\lambda Oh^{\mathrm{T}}(\delta\psi)=\lambda O$, the rotation $h(\delta\psi)$ belongs to $SA(2)$. Therefore, $L_{h(\delta\psi)}^{-1}=\exp(-\delta\psi\sigma^L(A_3))$. Calculating $\mathrm{d}\phi/\mathrm{d}\varepsilon$, $\mathrm{d}\theta/\mathrm{d}\varepsilon$ and $\mathrm{d}\psi/\mathrm{d}\varepsilon$ with equation (8.3), we have

$$\sigma(\hat{A}_1)=\sin\phi\frac{\partial}{\partial\theta}+\cot\theta\cos\phi\frac{\partial}{\partial\phi}+\frac{\cos\phi}{\sin\theta}\sigma^L(A_3) \tag{8.4a}$$

$$\sigma(\hat{A}_2)=-\cos\phi\frac{\partial}{\partial\theta}+\cot\theta\cos\phi\frac{\partial}{\partial\phi}+\frac{\cos\phi}{\sin\theta}\sigma^L(A_3) \tag{8.4b}$$

$$\sigma(\hat{A}_3)=-\frac{\partial}{\partial\phi}. \tag{8.4c}$$

If S is one of S_1^+, S_2^+, S_3^+ and S_4, we need to take $h(\phi)g(\theta)h(\psi)$ as $r \in SO(3)$. From

$$\mathrm{e}^{-\varepsilon A_k}h(\phi)g(\theta)h(\psi)=h(\phi+\delta\phi)g(\theta+\delta\theta)h(\psi+\delta\psi)$$

we can calculate $d\phi/d\varepsilon$, $d\theta/d\varepsilon$ and $d\psi/d\varepsilon$, to obtain

$$\sigma(\hat{A}_1)=\sin\phi\frac{\partial}{\partial\theta}+\cot\theta\cos\phi\frac{\partial}{\partial\phi}-\frac{\cos\phi}{\sin\theta}\frac{\partial}{\partial\psi} \tag{8.5a}$$

$$\sigma(\hat{A}_2)=-\cos\phi\frac{\partial}{\partial\theta}+\cot\theta\cos\phi\frac{\partial}{\partial\phi}-\frac{\cos\phi}{\sin\theta}\frac{\partial}{\partial\psi} \tag{8.5b}$$

$$\sigma(\hat{A}_3)=-\frac{\partial}{\partial\phi}. \tag{8.5c}$$

8.2. Symmetric tensor operators

Instead of calculating the expressions of $\sigma(\hat{E}_\mu)$ directly at Q, it is easier to calculate them at $Q'=r^{-l}\cdot Q$ and the origin $O=g_Q^{-l}\cdot Q$, and later transform them [36] to Q. If $Z\in sl(3,\mathbb{R})$ and $h\in Sl(3,\mathbb{R})$, then $hZh^{-1}\in sl(3,\mathbb{R})$. That is,

$$hZh^{-1}=\sum_{i=1}^{3}c_iA_i+\sum_{\mu=-2}^{2}d\mu\,E_\mu \qquad c_i\in\mathbb{R} \qquad d_\mu=(-1)^\mu d_{-\mu}\in\mathbb{C}.$$

Let $g_\varepsilon=h\,e^{\varepsilon Z}h^{-1}$. Since $h\,e^{\varepsilon Z}h^{-1}=\exp(\varepsilon hZh^{-1})$,

$$U_{g_\varepsilon}=\exp\left[\varepsilon\left(\sum_{i=1}^{3}c_i\hat{A}_i+\sum_{\mu=-2}^{2}d_\mu\hat{E}_\mu\right)\right].$$

Therefore,

$$\begin{aligned}\lim_{\varepsilon\to 0}\frac{(\rho(U_{g_\varepsilon})f)(Q)-f(Q)}{\varepsilon}&=\sigma\left(\sum_{i=1}^{3}c_i\hat{A}_i+\sum_{\mu=-2}^{2}d_\mu\hat{E}_\mu\right)f(Q)\\&=\left[\sum_{i=1}^{3}c_i\sigma(\hat{A}_i)+\sum_{\mu=-2}^{2}d_\mu\,\sigma(\hat{E}_\mu)\right]f(Q).\end{aligned} \tag{8.6}$$

The left side of the above equation can be calculated with equation (6.2). Since $\sigma(\hat{A}_k)$s are known, we can find $\sigma(\hat{E}_\mu)$s from five linear equations.

Let $g_\varepsilon=r\,e^{\varepsilon E_0}r^{-1}$. Then

$$f(g_\varepsilon^{-1}\cdot Q)\approx f(Q)-\varepsilon\frac{\partial f(Q)}{\partial\varepsilon_0}$$

$$\begin{aligned}\sqrt{\frac{d\mu(g_\varepsilon^{-1}\cdot Q)}{d\mu(Q)}}&\approx 1-\sqrt{6}\varepsilon &&\text{if } Q\in S_5^+\\&\approx 1+\frac{5}{\sqrt{6}}\varepsilon &&\text{if } Q\in S_3^+ \text{ or } S_4\end{aligned}$$

$\sqrt{d\mu_{1,2}(g^{-1}\cdot Q)/d\mu_{1,2}(Q)}=1$ for any $g\in SL(3,\mathbb{R})$. Since $rE_0r^{-1}=\sum_{\nu=-2}^{2}D_{\nu,0}^2E_\nu$ equation (8.6) becomes

$$\begin{aligned}\left[\sum_{\nu=-2}^{2}D_{\nu,0}^2\sigma(\hat{E}_\nu)\right]f(Q)&=-\left(\sqrt{6}+\frac{\partial}{\partial\varepsilon_0}\right)f(Q) &&\text{if } Q\in S_5^+\\&=\left(\frac{5}{\sqrt{6}}-\frac{\partial}{\partial\varepsilon_0}\right)f(Q) &&\text{if } Q\in S_3^+ \text{ or } S_4\\&=-\frac{\partial f(Q)}{\partial\varepsilon_0} &&\text{if } Q\in S_1^+ \text{ or } S_2^+.\end{aligned}$$

Let $g_\varepsilon = r\,e^{\varepsilon H_2}\,r^{-1}$. If $Q\in S_5^+$, then $g_\varepsilon^{-1}g_Q = r\lambda^{1/2}\,e^{-\varepsilon H_2}$ and $g_\varepsilon^{-1}\cdot Q = Q$. Consequently, the measure is unchanged by g_ε and $k = e^{-\varepsilon H_2}$ belongs to $SA(2)$. Therefore, $L_k^{-1} = \exp[\varepsilon\sigma^L(H_2)]$. If Q belongs to the other orbits,

$$f(g_\varepsilon^{-1}\cdot Q)\approx f(Q)-\varepsilon\frac{\partial f(Q)}{\partial\varepsilon_2}$$

$$\sqrt{\frac{\mathrm{d}\mu(g_\varepsilon^{-1}\cdot Q)}{\mathrm{d}\mu(Q)}}\approx 1-\varepsilon\frac{\lambda_1+\lambda_2}{\lambda_1-\lambda_2} \qquad \text{if } Q\in S_3^+$$

$$\approx 1-\varepsilon\frac{\lambda_1-\lambda_2}{\lambda_1+\lambda_2} \qquad \text{if } Q\in S_4$$

where $\lambda_1 = \exp(\sqrt{2}\varepsilon_2-\sqrt{2/3}\varepsilon_0)$ and $\lambda_2 = \exp(-\sqrt{2}\varepsilon_2-\sqrt{2/3}\varepsilon_0)$. Since $H_2 = (E_2+E_{-2})/\sqrt{2}$, equation (8.6) implies that

$$\begin{aligned}\left[\frac{1}{\sqrt{2}}\sum_{\nu=-2}^{2}(D^2_{\nu,2}+D^2_{\mu,-2})\sigma(\hat{E}_\nu)\right]f(Q)\\ &= \sigma^L(H_2)f(Q) \qquad \text{if } Q\in S_5^+\\ &= -\left(\frac{\partial}{\partial\varepsilon_2}+\frac{\lambda_1\pm\lambda_2}{\lambda_1\mp\lambda_2}\right)f(Q) \qquad \text{if } Q\in S_3^+ \text{ or } S_4\\ &= -\frac{\partial f(Q)}{\partial\varepsilon_2} \qquad \text{if } Q\in S_1^+ \text{ or } S_2^+.\end{aligned}$$

If $Z\in k$ and $g_\varepsilon = r\lambda^{1/2}\exp(\varepsilon Z)\lambda^{-1/2}r^{-1}$, then $g_\varepsilon^{-1}g_Q = g_Q\exp(-\varepsilon Z)$. Therefore, $\sqrt{\mathrm{d}\mu(g_\varepsilon^{-1}\cdot Q)/\mathrm{d}\mu(Q)} = 1$ and the right side of equation (8.6) is $\sigma^L(Z)f(Q)$.

If $Z = H_2$ then $g_\varepsilon = r\exp(\varepsilon H_2)r^{-1}$, and the expression of the left side of equation (8.6) is already known. We need to calculate the left side of equation (8.6) for A_is, B_is. The other elements of $\boldsymbol{k}$ are their linear combinations. Let

$$c_i(\lambda) = \tfrac{1}{2}\left(\sqrt{\frac{\lambda_k}{\lambda_j}}+\sqrt{\frac{\lambda_j}{\lambda_k}}\right) \qquad s_i(\lambda) = \tfrac{1}{2}\left(\sqrt{\frac{\lambda_k}{\lambda_j}}-\sqrt{\frac{\lambda_j}{\lambda_k}}\right) \qquad i = 1, 2, 3 \tag{8.7}$$

where i, j, k is a cyclic permutation of 1, 2, 3. Coefficients c_i and d_ν in equation (8.6) are calculated from the following formulae:

$$\lambda^{1/2}A_k\lambda^{-1/2} = s_k(\lambda)B_k + c_k(\lambda)A_k \qquad \lambda^{1/2}B_k\lambda^{-1/2} = s_k(\lambda)A_k + c_k(\lambda)B_k$$

and

$$B_1 = -\mathrm{i}(E_1+E_{-1}) \qquad B_2 = (E_{-1}-E_1) \qquad B_3 = \mathrm{i}(E_2-E_{-2}).$$

The expressions for angular momentum operators in the body fixed frame also differ according as $Q\in S_5^+$ or not. If r_{ij} denotes the (i, j) component of $r\in SO(3)$, then $rA_kr^{-1} = \Sigma_{j=1}^3 r_{jk}A_j$, and therefore $-\mathrm{i}\mathscr{K}_k = \sigma(\Sigma_{j=1}^3 r_{jk}\hat{A}_j) = \Sigma_{j=1}^3 r_{jk}\sigma(\hat{A}_j)$. If $r = h(\phi)g(\theta)h(\psi)$, we have the well known expressions

$$-\mathrm{i}\mathscr{K}_1 = -\cot\theta\cos\psi\frac{\partial}{\partial\psi}-\sin\psi\frac{\partial}{\partial\theta}+\frac{\cos\phi}{\sin\theta}\frac{\partial}{\partial\phi} \tag{8.8a}$$

$$-\mathrm{i}\mathscr{K}_2 = \cot\theta\sin\psi\frac{\partial}{\partial\psi}-\cos\psi\frac{\partial}{\partial\theta}-\frac{\sin\phi}{\sin\theta}\frac{\partial}{\partial\phi} \tag{8.8b}$$

$$-\mathrm{i}\mathscr{K}_3 = -\frac{\partial}{\partial\psi}. \tag{8.8c}$$

However, if $Q \in S_5^+$, since $r = h(\phi)g(\theta)$,

$$-\mathrm{i}\mathscr{K}_1 = \frac{1}{\sin\theta}\frac{\partial}{\partial\phi} + \cot\theta\,\sigma^L(A_3) \qquad -\mathrm{i}\mathscr{K}_2 = -\frac{\partial}{\partial\theta} \qquad -\mathrm{i}\mathscr{K}_3 = \sigma^L(A_3). \tag{8.9}$$

With the above expressions, expressions for $\sigma(\hat{E}_\mu)$s are calculated. All of them are of the form $\sigma(\hat{E}_\mu) = \Sigma_{\nu=-2}^{2} \bar{D}^2_{\mu,\nu}(\phi, \theta, \psi) T_\nu$, where ψ is taken to be identically 0 on S_5^+. We list the expressions of T_νs below.

On S_5^+,

$$T_{\pm 2} = \tfrac{1}{2}[\sqrt{2}\sigma^L(H_2) \mp \mathrm{i}\sigma^L(B_3)] \tag{8.10a}$$

$$T_{\pm 1} = \tfrac{1}{2}\left[\mathrm{e}^{-\sqrt{6}\varepsilon_0}\{\mathrm{i}\sigma^L(Z_1^{\mathrm{T}}) \mp \sigma^L(Z_2^{\mathrm{T}})\}\right.$$

$$\left.\mp\frac{\partial}{\partial\theta} - \frac{\mathrm{i}}{\sin\theta}\left(\frac{\partial}{\partial\phi} + \cot\theta\,\sigma^L(A_3)\right)\right] \tag{8.10b}$$

$$T_0 = -\left(\sqrt{6} + \frac{\partial}{\partial\varepsilon_0}\right). \tag{8.10c}$$

On S_3^+,

$$T_{\pm 2} = \tfrac{1}{2}\left[\sqrt{2}\left(-\frac{\partial}{\partial\varepsilon_2} - \frac{\lambda_1+\lambda_2}{\lambda_1-\lambda_2}\right) \mp \left(\mathrm{i}\frac{\sigma^L(A_3)}{s_3(\lambda)} - \frac{c_3(\lambda)}{s_3(\lambda)}\mathscr{K}_3\right)\right] \tag{8.11a}$$

$$T_{\pm 1} = \tfrac{1}{2}\left[\mathrm{i}\frac{\lambda_3}{\lambda_1}\sigma^L(Z_1) \mp \frac{\lambda_3}{\lambda_2}\sigma^L(Z_2) + \mathscr{K}_1 \mp \mathrm{i}\mathscr{K}_2\right] \tag{8.11b}$$

$$T_0 = \left(\frac{5}{\sqrt{6}} - \frac{\partial}{\partial\varepsilon_0}\right) \tag{8.11c}$$

where $\lambda_1 = \exp(\sqrt{2}\varepsilon_2 - \sqrt{2/3}\varepsilon_0)$, $\lambda_2 = \exp(-\sqrt{2}\varepsilon_2 - \sqrt{2/3}\varepsilon_0)$.

On S_4, the expressions of T_0 and $T_{\pm 1}$ are the same as above, and

$$T_{\pm 2} = \tfrac{1}{2}\left[\sqrt{2}\left(-\frac{\partial}{\partial\varepsilon_2} - \frac{\lambda_1-\lambda_2}{\lambda_1+\lambda_2}\right) \mp \left(\mathrm{i}\frac{\sigma^L(B_2)}{s_3(\lambda)} - \frac{s_3(\lambda)}{c_3(\lambda)}\mathscr{K}_3\right)\right]. \tag{8.12}$$

On S_1^+

$$T_{\pm 2} = \tfrac{1}{2}\left[-\sqrt{2}\frac{\partial}{\partial\varepsilon_2} \mp \left(\mathrm{i}\frac{\sigma^L(A_3)}{s_3(\lambda)} - \frac{c_3(\lambda)}{s_3(\lambda)}\mathscr{K}_3\right)\right] \tag{8.13a}$$

$$T_{\pm 1} = \tfrac{1}{2}\left[\mathrm{i}\frac{\sigma^L(A_1)}{s_1(\lambda)} \mp \frac{\sigma^L(A_2)}{s_2(\lambda)} - \frac{c_1(\lambda)}{s_1(\lambda)}\mathscr{K}_1 \mp \frac{c_2(\lambda)}{s_2(\lambda)}\mathrm{i}\mathscr{K}_2\right] \tag{8.13b}$$

$$T_0 = -\frac{\partial}{\partial\varepsilon_0}. \tag{8.13c}$$

On S_2^+, the expressions of T_0 and $T_{\pm 2}$ are the same as above, and

$$T_{\pm 1} = \tfrac{1}{2}\left[\mathrm{i}\frac{\sigma^L(B_1)}{c_1(\lambda)} \mp \frac{\sigma^L(B_2)}{c_2(\lambda)} - \frac{s_1(\lambda)}{c_1(\lambda)}\mathscr{K}_1 \mp \frac{s_2(\lambda)}{c_2(\lambda)}\mathrm{i}\mathscr{K}_2\right]. \tag{8.14}$$

Transforming variables from ε_0 and ε_2 to $v=\sqrt{\lambda_1\lambda_2}$ and $\varepsilon=\log\sqrt{\lambda_1/\lambda_2}$ or $\rho=\mathrm{e}^{\varepsilon}-\mathrm{e}^{-\varepsilon}$ is straightforward.

9. *CM*(3) model Hamiltonian

If $Q\in S_1^+$ and $\beta=\sqrt{\varepsilon_0^2+\varepsilon_2^2}$ is small and the representation of $SO(3)$ is trivial, $\sigma(\hat{E}_\mu)$ approaches to $-\partial/\partial\alpha_\mu = -\Sigma_{\nu=-2}^{2}\bar{D}^2_{\mu,\nu}\pi_\nu$, where α_μ is given in equation (1.11) and

$$\pi_{\pm 2} = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial\varepsilon_2} \pm \frac{\mathscr{K}_3}{2\varepsilon_2}\right) \qquad \pi_{\pm 1} = \tfrac{1}{2}\left(\frac{\mathscr{K}_1}{u+v} \mp \mathrm{i}\frac{\mathscr{K}_2}{u-v}\right) \qquad \pi_0 = \frac{\partial}{\partial\varepsilon_0} \tag{9.1}$$

where $u=\sqrt{3/2}\,\varepsilon_0$ and $v=\varepsilon_2/\sqrt{2}$.

Therefore, the analogue of the Bohr Hamiltonian in the *CM*(3) model will be

$$\mathscr{H} = -\frac{1}{2B}\sum_{\mu=-2}^{2}(-1)^\mu\hat{\sigma}(E_\mu)\hat{\sigma}(E_{-\mu}) + C\sum_{\mu=-2}^{2}(-1)^\mu Q_\mu^2 Q_{-\mu}^2 \tag{9.2}$$

where B and C are parameters which cannot be determined from the model.

The reason why only the representation of $cm(3)$ in $\mathscr{L}^2(S_1^+,\mu_1)$ has the correspondence with the Bohr model is as follows. Consider a quadratic form $A(y)=\Sigma_{i,j=1}^3 Q_{ij}y_iy_j$, where $y=(y_1,y_2,y_3)$. With formula (1.1), $A(y)=\Sigma_{n=1}^{A}(\Sigma_{i=1}^{3}x_{in}y_i)^2$. Let $\Psi(\mathbf{x}_1,\ldots,\mathbf{x}_A)$ be a wavefunction of the A-particle system, $\mathrm{d}\mathbf{x}_n=\mathrm{d}x_{1n}\,\mathrm{d}x_{2n}\,\mathrm{d}x_{3n}$ and

$$\rho_n(\mathbf{x}_n) = \int_{\mathbb{R}^{3(A-1)}} \bar{\Psi}(\mathbf{x}_1,\ldots,\mathbf{x}_A)\Psi(\mathbf{x}_1,\ldots,\mathbf{x}_A)\,\mathrm{d}\mathbf{x}_1\ldots\mathrm{d}\mathbf{x}_{n-1}\,\mathrm{d}\mathbf{x}_{n+1}\ldots\mathrm{d}\mathbf{x}_A.$$

Then $\langle\Psi|A(y)|\Psi\rangle=\Sigma_{n=1}^{A}\int(\Sigma_{i=1}^{3}x_{in}y_i)^2\rho_n(\mathbf{x}_n)\,\mathrm{d}\mathbf{x}_n\geqslant 0$. Therefore, if $\langle\Psi|A(y)|\Psi\rangle=0$, then $\int(\Sigma_{i=1}^{3}x_{in}y_i)^2\rho_n(\mathbf{x}_n)\,\mathrm{d}\mathbf{x}_n=0$ for any n. As $\rho_n(\mathbf{x}_n)\geqslant 0$ and is not identically 0, $(\Sigma_{i=1}^{3}x_{in}y_i)^2=0$ for any n. As y is arbitrary, $\mathbf{x}_n=0$ for any n. Thus, the matrix (Q_{ij}) must be positive definite. Only the representation in $\mathscr{L}^2(S_1^+,\mu_1)$ satisfies the condition. We do not know if the other representations have applications in some field of physics or not.

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